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I am submitting herewith a thesis written by Kristi Dawn Renninger entitled "Lattice Paths in $\mathbb{R}^2$ and $\mathbb{R}^3$." I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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LATTICE PATHS IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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ABSTRACT

The focus in this thesis is on lattice paths in $\mathbb{R}^2$ and $\mathbb{R}^3$. Three ways of counting the number of lattice paths from the origin to a point in $\mathbb{R}^2$ and $\mathbb{R}^3$ are examined: the recurrence relation, the closed form expression, and the generating function. In the first two chapters, lattice paths from $(0, 0)$ to (p, q) are discussed, first without restriction and then with restrictions based on whether the path may or may not cross or touch the line $y = x$. In the first chapter, lattice paths are not allowed to move diagonally, and it is shown how the ballot problem is an application. In the second chapter, lattice paths are allowed diagonal moves. In the third and fourth chapters, lattice paths from $(0, 0, 0)$ to (p, q, r) are also discussed first without restriction and then with the restrictions that for each point (i, j, k) on the lattice path $i \geq j$, $i > j$, or $i \geq j \geq k$. In the third chapter, diagonal moves are not allowed, and the ballot problem is discussed. In the fourth chapter, diagonal moves are allowed.

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Chapter 1 Lattice Paths in $\mathbb{R}^2$

1.1. Unrestricted Paths: $L(p, q)$

Unrestricted lattice paths from the point $(0, 0)$ to the point (p, q) consist of some sequential arrangement of the p moves east and q moves north necessary to reach the point (p, q) . The total number of these lattice paths is denoted by $L(p, q)$. There are several different ways to count these paths. One method is by using the recursive formula which establishes the initial conditions and a formula based on previous values of $L(p, q)$. The following is a recursive formula for $L(p, q)$.

Theorem 1.1: For all $p, q \in \mathbb{N}$, $L(p, 0) = 1$ and $L(0, q) = 1$. For all $p, q \in \mathbb{P}$,

$$L(p, q) = L(p, q - 1) + L(p - 1, q). \quad (1.1)$$

Proof: The only lattice path from $(0, 0)$ to $(p, 0)$ is the one that consists of p E's. Thus, $L(p, 0) = 1$. Similarly, the only path from $(0, 0)$ to $(0, q)$ is the one that consists of q N's, so $L(0, q) = 1$.

For 1.1, the LHS counts the number of lattice paths from $(0, 0)$ to (p, q) by definition. The RHS counts this number in two disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q) is $(p, q - 1)$,
- ii. The category in which the last point visited before (p, q) is $(p - 1, q)$. ■

From the recursive formula, we can generate Table 1.1.

$L(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	1	1	1	1
	1	1	2	3	4	5
	2	1	3	6	10	15
	3	1	4	10	20	35
	4	1	5	15	35	70

Table 1.1

Another method that we can use to determine $L(p, q)$ is to find a closed form expression which requires only one calculation rather than the repeated calculations of the recursive formula.

Theorem 1.2: For all $p, q \in \mathbb{N}$,

$$L(p, q) = \binom{p+q}{p} = \binom{p+q}{q}.$$

Proof: Any lattice path from $(0, 0)$ to (p, q) consists of p moves east and q moves north.

Thus, the set of such lattice paths is in one-to-one correspondence with the set of words of length $p + q$ consisting of p E's and q N's. The p positions to be occupied by E's can be chosen from the $p + q$ positions, or, equivalently, the q positions to be occupied by N's can be chosen. Thus, the number of words of

$$\text{length } p + q \text{ is } \binom{p+q}{p} \text{ or } \binom{p+q}{q}. \quad \blacksquare$$

In this thesis, we will also establish the generating function for some of the types of lattice paths that we examine. We will use three methods of finding the generating function for unrestricted lattice paths. The first two ways use the closed form expression. The third way uses the recursive formula. To find the generating function using the closed form expression, we must first establish the next two results, which will be used without explicit reference throughout this thesis. Newton's binomial theorem is the following:

Theorem 1.3: For all $\alpha \in \mathbb{R}$ and $|x| < 1$,

$$(1+x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k.$$

Proof: Using the Maclaurin series expansion of $(1+x)^\alpha$ for $|x| < 1$,

$$(1+x)^\alpha = \sum_{k \geq 0} \frac{\left. \frac{d^k}{dx^k} (1+x)^\alpha \right|_{x=0}}{k!} x^k$$

$$\begin{aligned}
&= \frac{1}{0!} x^0 + \frac{\alpha}{1!} x^1 + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \dots \\
&= \sum_{k \geq 0} \binom{\alpha}{k} x^k. \quad \blacksquare
\end{aligned}$$

Theorem 1.4: For all $k \in \mathbb{N}$,

$$\frac{1}{(1-x)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{n} x^n.$$

Proof: Let $\alpha = -(k+1)$ in Theorem 1.3. Then,

$$\begin{aligned}
\frac{1}{(1-x)^{k+1}} &= (1-x)^{-(k+1)} = \sum_{n \geq 0} \binom{-(k+1)}{n} (-x)^n \\
&= \sum_{n \geq 0} (-1)^n \binom{-k-1}{n} x^n = \sum_{n \geq 0} (-1)^n (-1)^n \frac{(k+1)(k+2) \cdots (k+n)}{n!} x^n \\
&= \sum_{n \geq 0} \frac{(n+k) \cdots (k+2)(k+1)k!}{n!k!} x^n = \sum_{n \geq 0} \binom{n+k}{n} x^n. \quad \blacksquare
\end{aligned}$$

Now, we are able to find the generating function for $L(p, q)$.

Theorem 1.5: The generating function for $L(p, q)$,

$$\sum_{p \geq 0} \sum_{q \geq 0} L(p, q) x^p y^q = \frac{1}{1-x-y}.$$

Proof 1:

$$\sum_{p \geq 0} \sum_{q \geq 0} L(p, q) x^p y^q = \sum_{p \geq 0} \sum_{q \geq 0} \binom{p+q}{q} x^p y^q$$

$$\begin{aligned}
&= \sum_{p \geq 0} x^p \sum_{q \geq 0} \binom{p+q}{q} y^q = \sum_{p \geq 0} x^p \frac{1}{(1-y)^{p+1}} \\
&= \frac{1}{1-y} \sum_{p \geq 0} \left(\frac{x}{1-y} \right)^p = \frac{1}{1-y} \frac{1}{1 - \frac{x}{1-y}} \\
&= \frac{1}{1-x-y}.
\end{aligned}$$

Proof 2:

$$\begin{aligned}
\sum_{p,q \geq 0} L(p,q) x^p y^q &= \sum_{p,q \geq 0} \binom{p+q}{p} x^p y^q \\
&= \sum_{n \geq 0} \sum_{p+q=n} \binom{n}{p} x^p y^q = \sum_{n \geq 0} (x+y)^n = \frac{1}{1-x-y}.
\end{aligned}$$

Proof 3: Let $F(x, y) = \sum_{p,q \geq 0} L(p,q) x^p y^q$. Then, by Theorem 1.1,

$$\begin{aligned}
F(x, y) &= \sum_{p \geq 0} L(p,0) x^p y^0 + \sum_{q \geq 0} L(0,q) x^0 y^q - L(0,0) x^0 y^0 \\
&\quad + x \sum_{p,q > 0} L(p-1,q) x^{p-1} y^q + y \sum_{p,q > 0} L(p,q-1) x^p y^{q-1} \\
&= \frac{1}{1-x} + \frac{1}{1-y} - 1 + x \sum_{\substack{p \geq 0 \\ q > 0}} L(p,q) x^p y^q + y \sum_{\substack{p > 0 \\ q \geq 0}} L(p,q) x^p y^q
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x} + \frac{1}{1-y} - 1 + x \left(\sum_{p,q \geq 0} L(p,q) x^p y^q - \sum_{p \geq 0} L(p,0) x^p y^0 \right) \\
&\quad + y \left(\sum_{p,q \geq 0} L(p,q) x^p y^q - \sum_{q \geq 0} L(0,q) x^0 y^q \right) \\
&= 1 + xF(x,y) + yF(x,y),
\end{aligned}$$

and so,

$$(1-x-y)F(x,y) = 1$$

$$F(x,y) = \frac{1}{1-x-y}. \quad \blacksquare$$

1.2. Paths That Do Not Rise Above the Diagonal: $L^*(p, q)$

We now want to consider the number of lattice paths from the point $(0, 0)$ to the point (p, q) that never rise above the diagonal $y = x$. This number will be denoted by $L^*(p, q)$. The following is a recursive formula for $L^*(p, q)$:

Theorem 1.6: For all $p \in \mathbf{N}$, $L^*(p, 0) = 1$. For all $p, q \in \mathbf{N}$ with $p < q$, $L^*(p, q) = 0$. For all $p, q \in \mathbf{P}$ with $p \geq q$,

$$L^*(p, q) = L^*(p, q-1) + L^*(p-1, q). \quad (1.2)$$

Proof: The only lattice path from $(0, 0)$ to $(p, 0)$ is the one that consists of p E's. Thus, $L^*(p, 0) = 1$. If $p, q \in \mathbf{N}$ and $p < q$, then $L^*(p, q) = 0$ because any lattice path from $(0, 0)$ to (p, q) contains the point (p, q) , which lies above the diagonal.

Suppose $p, q \in \mathbf{P}$ and $p \geq q$. Then, by definition, the LHS of 1.2 counts the set of lattice paths that never move above the diagonal. The RHS of 1.2 counts this set in two disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q) is $(p, q-1)$, and
- ii. The category in which the last point visited before (p, q) is $(p-1, q)$. If $p = q$, this category is empty, so $L(p, p) = L(p, p-1)$ for all $p \in \mathbf{P}$. $\blacksquare$

Several values of $L^*(p, q)$ are given in Table 1.2.

$L^*(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	0	0	0	0
	1	1	1	0	0	0
	2	1	2	2	0	0
	3	1	3	5	5	0
	4	1	4	9	14	14

Table 1.2

Theorem 1.7: For all $p, q \in \mathbb{N}$,

$$L^*(p, q) = \begin{cases} 0 & \text{if } p < q \\ \frac{p - q + 1}{p + 1} \binom{p + q}{q} & \text{if } p \geq q \end{cases}$$

Proof: If $p < q$, $L^*(p, q) = 0$ by Theorem 1.6.

Suppose $p \geq q$. By Theorem 1.2, the total number of lattice paths from

$(0, 0)$ to (p, q) is $\binom{p + q}{q}$. Now the number of lattice paths that cross the

diagonal must be determined. Such a lattice path first crosses the diagonal at a certain point, say (k, k) . Thus, the point $(k, k + 1)$ is a point on this lattice path, so the path has k E's and $k + 1$ N's up to this point. The rest of the path consists of $p - k$ E's and $q - k - 1$ N's. Convert the $p - k$ E's to N's and the $q - k - 1$ N's to E's, leaving the initial part of the path unchanged. The result is a path from $(0, 0)$ to $(q - 1, p + 1)$, and the map just defined is in fact a bijection from the set of all lattice paths from $(0, 0)$ to (p, q) that rise above the diagonal to the set of all lattice paths from $(0, 0)$ to $(q - 1, p + 1)$. For the inverse of this map is obvious: Each lattice path from $(0, 0)$ to $(q - 1, p + 1)$ must visit a point $(k, k + 1)$. It may touch more than one point of this type, so we want to consider where the lattice path

first visits such a point. At this point, we change the remaining $q - k - 1$ N's to E's and $p - k$ E's to N's, and we have a lattice path that crosses the line $y = x$ with

p N's and q E's. By Theorem 1.2, there are thus $\binom{p+q}{q-1}$ paths from $(0, 0)$ to

(p, q) which rise above the diagonal, and so

$$L^*(p, q) = \binom{p+q}{q} - \binom{p+q}{q-1}$$

$$= \binom{p+q}{q} - \frac{q}{p+1} \binom{p+q}{q} = \frac{p-q+1}{p+1} \binom{p+q}{q}. \quad \blacksquare$$

From this theorem, it follows that $L^*(p, 1) = p$ and $L^*(p, 2) = \frac{1}{2}(p+2)(p-1)$. In particular, $L^*(p, p) = \frac{1}{p+1} \binom{2p}{p}$ which is the p th Catalan number. These numbers

arise in many combinatorial problems.

It is difficult to establish a generating function for $L^*(p, q)$; however, we can establish the following partial results.

Theorem 1.8: For $p, q \in \mathbb{N}$,

- i. $\sum_{p \geq 0} L^*(p, 0) x^p = \frac{1}{1-x},$
- ii. $\sum_{p \geq 0} L^*(p, 1) x^p = \frac{x}{(1-x)^2},$

$$\text{iii. } \sum_{p \geq 0} L^*(p, 2)x^p = \frac{-1 + 3x - x^2}{(1-x)^3}, \text{ and}$$

$$\text{iv. } \sum_{p \geq 0} L^*(p, p)x^p = \frac{1 - \sqrt{1-4x}}{2x}.$$

Proof:

i. By Theorem 1.8,

$$\sum_{p \geq 0} L^*(p, 0)x^p = \sum_{p \geq 0} 1x^p = \sum_{p \geq 0} x^p = \frac{1}{1-x}.$$

ii. By Theorem 1.7,

$$\sum_{p \geq 0} L^*(p, 1)x^p = \sum_{p \geq 0} px^p = \sum_{p \geq 0} \binom{p}{1} x^p = \frac{x}{(1-x)^2}.$$

iii. By Theorem 1.7,

$$\begin{aligned} \sum_{p \geq 0} L^*(p, 2)x^p &= \sum_{p \geq 0} \frac{1}{2}(p^2 + p - 2)x^p \\ &= \frac{1}{2} \sum_{p \geq 0} p^2 x^p + \frac{1}{2} \sum_{p \geq 0} px^p - \sum_{p \geq 0} x^p \\ &= \frac{1}{2} \sum_{p \geq 0} \left[2 \binom{p}{2} + \binom{p}{1} \right] x^p + \frac{1}{2} \sum_{p \geq 0} \binom{p}{1} x^p - \sum_{p \geq 0} x^p \\ &= \sum_{p \geq 0} \binom{p}{2} x^p + \sum_{p \geq 0} \binom{p}{1} x^p - \sum_{p \geq 0} x^p = \frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2} - \frac{1}{1-x} \end{aligned}$$

$$= \frac{x^2 + x - x^2 - (1-x)^2}{(1-x)^3} = \frac{-1 + 3x - x^2}{(1-x)^3}$$

iv. By using Newton's Binomial Theorem, we obtain

$$\begin{aligned} \frac{1 - \sqrt{1-4x}}{2x} &= \frac{1 - \sum_{k \geq 0} \binom{1/2}{k} (-4x)^k}{2x} = \frac{1 - 1 - \sum_{k \geq 1} \binom{1/2}{k} (-4x)^k}{2x} \\ &= \frac{-\sum_{k \geq 1} \binom{1/2}{k} (-1)^k 4^k x^k}{2x} = -\frac{1}{2} \sum_{k \geq 1} \binom{1/2}{k} (-1)^k 4^k x^{k-1} \end{aligned}$$

Let $n = k - 1$.

$$= \sum_{n \geq 0} \frac{1}{2} \binom{1/2}{n+1} (-1)^n 4^{n+1} x^n.$$

Now, we want to consider the coefficient.

$$\begin{aligned} \frac{1}{2} \binom{1/2}{n+1} (-1)^n 4^{n+1} &= \frac{1}{2} (-1)^n 4^{n+1} \frac{\frac{1}{2} (-\frac{1}{2}) (-\frac{3}{2}) \cdots (\frac{1}{2} - n)}{(n+1)!} \\ &= \frac{1}{2} (-1)^n 4^{n+1} \left(\frac{1}{2}\right)^{n+1} (-1)^n \frac{(1)(1)(3) \cdots (2n-1)}{(n+1)!} = 2^n \frac{(1)(3) \cdots (2n-1)}{(n+1)!} \\ &= 2^n \frac{(1)(2)(3) \cdots (2n)}{2^n n!(n+1)!} = \frac{1}{n+1} \binom{2n}{n}. \quad \blacksquare \end{aligned}$$

An application of Theorem 1.7 occurs in the ballot problem. In an election with two candidates, candidate A receives p votes and candidate B receives q votes, where $p \geq q$. We want to determine the probability that if the votes are recorded randomly the partial tally at every stage shows that Candidate A does not trail Candidate B.

Theorem 1.9: For $p \geq q \geq 0$, the probability that Candidate A never trails Candidate B is

$$\frac{p - q + 1}{p + 1}.$$

Proof: We need to determine the number of ways to arrange the $p + q$ votes such that the number of votes that candidate A has received at any point in the election is always more than or equal to the number of votes that candidate B has received. Representing a vote for A as an E and a vote for B as an N, we see that the solution to this problem requires determining the number of lattice paths from $(0, 0)$ to (p, q) such that the path never crosses the diagonal $y = x$. By Theorems 1.2 and 1.7, the probability that Candidate A never trails Candidate B is

$$\frac{\frac{p - q + 1}{p + 1} \binom{p + q}{q}}{\binom{p + q}{q}} = \frac{p - q + 1}{p + 1}. \quad \blacksquare$$

1.3. Paths Strictly Below the Diagonal Except at $(0, 0)$: $L^{**}(p, q)$

We now want to consider lattice paths that are not allowed to cross or touch the diagonal $y = x$ except at the point $(0, 0)$. The number of such paths will be denoted $L^{**}(p, q)$.

Theorem 1.10: For all $p \in \mathbb{N}$, $L^{**}(p, 0) = 1$. If $q \in \mathbb{P}$, $p \in \mathbb{N}$, and $p \leq q$,

$L^{**}(p, q) = 0$. For $p, q \in \mathbb{P}$ with $p > q$,

$$L^{**}(p, q) = L^{**}(p - 1, q) + L^{**}(p, q - 1). \quad (1.3)$$

Proof: From $(0, 0)$ to $(p, 0)$, there only path is the one that consists of p E's. Thus,

$L^{**}(p, 0) = 1$. Suppose $p \leq q$. Then, any lattice path to (p, q) must either touch or cross the diagonal to reach the point. Hence, $L^{**}(p, q) = 0$ if $p \leq q$.

Suppose $p > q$. The LHS of 1.3 counts the number of lattice paths that neither cross nor touch the diagonal except at $(0, 0)$ by definition. The RHS of 1.3

counts this set in two disjoint exhaustive categories:

- ii The category in which the last point visited before (p, q) is $(p - 1, q)$. If $p - 1 = q$, then this category is empty, and
- ii The category in which the last point visited before (p, q) is $(p, q - 1)$. ■

We can use the recursive formula to generate Table 1.3.

$L^{**}(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	0	0	0	0
	1	1	0	0	0	0
	2	1	1	0	0	0
	3	1	2	2	0	0
	4	1	3	5	5	0

Table 1.3

Theorem 1.11: If $p > q \geq 0$,

$$L^{**}(p, q) = L^*(p - 1, q) = \frac{p - q}{p + q} \binom{p + q}{q}.$$

Proof: Any lattice path in this set must visit the point $(1, 0)$ first. Thus, we can remove an E from the sequence of E's and N's. This is the same as finding the number of lattice paths from $(0, 0)$ to $(p - 1, q)$ which do not cross the diagonal $y = x$. Hence,

$$\begin{aligned} L^{**}(p, q) &= L^*(p - 1, q) = \frac{(p - 1) - q + 1}{(p - 1) + 1} \binom{p - 1 + q}{q} \\ &= \frac{p - q}{p} \binom{p + q - 1}{q} = \frac{p - q}{p} \frac{p + q}{p + q} \binom{p + q - 1}{q} = \frac{p - q}{p + q} \binom{p + q}{q}. \quad \blacksquare \end{aligned}$$

This result can be applied to the ballot problem where candidate A receives p votes and candidate B receives q votes where $p > q$. We want to determine the probability that if the votes are recorded randomly the partial tally at every stage correctly

predicts the result that candidate A wins.

Theorem 1.12: For $p > q \geq 0$, the probability that candidate A always has more

votes than candidate B is $\frac{p-q}{p+q}$.

Proof: If we count each vote in the order that it was cast, it is the same as counting the number of E moves and N moves in a lattice path from $(0, 0)$ to (p, q) . The total

number of lattice paths from $(0, 0)$ to (p, q) is $\binom{p+q}{q}$ by Theorem 1.2. The

total number of lattice paths from $(0, 0)$ to (p, q) where p is always greater than q

is $\frac{p-q}{p+q} \binom{p+q}{q}$ by Theorem 1.11. Thus, the probability that candidate A

always has more votes than candidate B at any point during the election is

$$\frac{\frac{p-q}{p+q} \binom{p+q}{q}}{\binom{p+q}{q}} = \frac{p-q}{p+q}. \quad \blacksquare$$

1.4. Paths Strictly Below the Diagonal

Except at $(0, 0)$ and at (p, q) if $p = q$: $L^{(**)}(p, q)$

Another set of lattice paths to consider is those in which the lattice path is allowed to touch the diagonal at only the points $(0, 0)$ and (p, q) if $p = q$. The number of such lattice paths will be denoted $L^{(**)}(p, q)$. We would prefer to find an easy recursive formula for $L^{(**)}(p, q)$ as we did with the previous three cases. However inspection of Table 1.4 shows the difficulty in finding such a formula.

$L^{(**)}(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	0	0	0	0
	1	1	1	0	0	0
	2	1	1	1	0	0
	3	1	2	2	2	0
	4	1	3	5	5	5

Table 1.4

If $q = 0$, then $L^{(**)}(p, 0) = 1$ because there is only one path from $(0, 0)$ to $(p, 0)$. Since the lattice path must cross the diagonal to reach (p, q) if $p < q$, then $L^{(**)}(p, q) = 0$. The difficulty arises when $q = p - 1$ for $q \geq 1$. In this case, it can be seen from the table that $L^{(**)}(p, p - 1) \neq L^{(**)}(p - 1, p - 1) + L^{(**)}(p, p - 2)$. The usual two term recurrence holds for $L(p, q)$ if $p - 2 \geq q \geq 1$, but the boundary conditions that accompany this recurrence are complicated to state. However, we can establish the following results about $L^{(**)}(p, q)$.

Theorem 1.13: For $p \in \mathbb{P}$, $q \in \mathbb{N}$, and $p > q$,

$$L^{(**)}(p, q) = L^{**}(p, q).$$

Proof: If $p > q$, then the point (p, p) is not a point that the lattice path can ever visit.

Thus, $L^{(**)}(p, q)$ and $L^{**}(p, q)$ are the number of lattice paths that never touch the diagonal $y = x$ except at $(0, 0)$. ■

Theorem 1.14: For $p \in \mathbb{P}$,

$$L^{(**)}(p, p) = L^{**}(p, p - 1).$$

Proof: By definition, $L^{**}(p, p - 1)$ is the number of lattice paths from $(0, 0)$ to $(p, p - 1)$ that do not touch the diagonal $y = x$ except at $(0, 0)$. Because $L^{(**)}(p, p)$ is the number of lattice paths that touch the diagonal only at $(0, 0)$ and (p, p) , all lattice paths from $(0, 0)$ to (p, p) must first visit the point $(p, p - 1)$ with the last move being the north move from $(p, p - 1)$ to (p, p) . ■

1.5 Notes

Theorems 1.1, 1.2, 1.3, 1.4, and their proofs are found in [7]. The source of Theorem 1.5 is Wagner [8]. Theorem 1.7 is found in [5]. The proof of Theorem 1.8 is

adapted from a similar proof in [7]. Theorem 1.11 is found in [5]. Theorem 1.12 is found in [5].

Chapter 2

Lattice Paths with Diagonal Moves in $\mathbb{R}^2$

2.1. Unrestricted Paths: $L_d(p, q)$, the “Delannoy numbers”

We will now consider lattice paths that allow diagonal steps. A diagonal step is a step from (x, y) to $(x + 1, y + 1)$. The number of lattice paths from $(0, 0)$ to (p, q) , with diagonal steps allowed, is denoted by $L_d(p, q)$. These numbers are called the Delannoy numbers. The following is a recursive formula for $L_d(p, q)$.

Theorem 2.1: For all $p, q \in \mathbb{N}$, $L_d(p, 0) = 1$ and $L_d(0, q) = 1$. For all $p, q \in \mathbb{P}$,

$$L_d(p, q) = L_d(p - 1, q) + L_d(p, q - 1) + L_d(p - 1, q - 1). \quad (2.1)$$

Proof: From the point $(0, 0)$ to the point $(p, 0)$, the only path is the one that consists of p E's, so $L_d(p, 0) = 1$. Also, the only path from $(0, 0)$ to $(0, q)$ is the path that consists of q N's, so $L_d(0, q) = 1$.

By definition, the LHS of 2.1 counts the set of lattice paths, with diagonals allowed, from $(0, 0)$ to (p, q) . The RHS counts this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q) is $(p - 1, q)$,
- ii. The category in which the last point visited before (p, q) is $(p, q - 1)$, and
- iii. The category in which the last point visited before (p, q) is $(p - 1, q - 1)$. ■

Several values for $L_d(p, q)$ are listed in Table 2.1.

$L_d(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	1	1	1	1
	1	1	3	5	7	9
	2	1	5	13	25	41
	3	1	7	25	63	129
	4	1	9	41	129	321

Table 2.1

Theorem 2.2: For all $p, q \in \mathbb{N}$, $L_d(p, q)$ is odd.

Proof: We prove this result by induction on $p + q$. If $p + q \leq 1$, then either $p = 0$ or $q = 0$,

so $L_d(p, q) = 1$.

Assume $L_d(a, b)$ is odd for all $a, b \in \mathbb{N}$ with $a + b < n$, where $n \geq 2$ and suppose that $p + q = n$. If $p = 0$ or $q = 0$, then $L_d(p, q) = 1$. If $p, q \in \mathbb{P}$, then by Theorem 2.1, $L_d(p, q) = L_d(p - 1, q) + L_d(p, q - 1) + L_d(p - 1, q - 1)$. By the induction hypothesis, each term of the RHS of the above equation is odd, and the sum of three odd numbers is odd. ■

We can find several closed form expressions for $L_d(p, q)$. Also, when $p = q$, $L_d(p, q)$ is related to the Legendre polynomial.

Theorem 2.3: For $p, q \in \mathbb{N}$,

- i. $L_d(p, 1) = 2p + 1$,
- ii. $L_d(p, 2) = 2p^2 + 2p + 1$,
- iii. $L_d(p, q) = \sum_{k=0}^{\min\{p, q\}} \binom{p+q-k}{k, p-k, q-k}$, and
- iv. $L_d(p, p) = P_n(3)$, the Legendre polynomial $P_n(x)$ at $x = 3$.

Proof:

- i. Between $(0, 0)$ and $(p, 1)$, a path with either a move north or a diagonal move is necessary. There are p paths that contain a diagonal move and $(p + 1)$ paths that contain a move north. Thus, $L_d(p, q) = 2p + 1$.
- ii. By Theorem 1.2, the number of lattice paths without diagonal moves

between $(0, 0)$ and $(p, 2)$ is $\binom{p+2}{2}$. If a path has one diagonal move,

there is a choice of which of two rows to connect with the diagonal moves, and there are $(p - 1)$ moves east, one move north, and one diagonal move.

Hence, there are $2 \binom{p+1}{p-1}$ paths with one diagonal move. If the lattice

path has two diagonal moves, the path has $(p - 2)$ moves east and no moves north. Thus, the number of paths with two diagonal moves is

$$\binom{p}{p-2}. \text{ Hence,}$$

$$\begin{aligned} L_d(p, q) &= \binom{p+2}{p} + 2\binom{p+1}{p-1} + \binom{p}{p-2} \\ &= \frac{(p+2)(p+1)}{2} + 2\frac{(p+1)p}{2} + \frac{p(p-1)}{2} = 2p^2 + 2p + 1. \end{aligned}$$

- iii. Suppose that there are k diagonal moves where $0 \leq k \leq \min\{p, q\}$. Then, there are $p - k$ moves east and $q - k$ moves north. The set of lattice paths with k diagonal moves allowed is in one-to-one correspondence with the set of words with k D's, $p - k$ E's, and $q - k$ N's which is given by the

$$\text{trinomial coefficient } \binom{p+q-k}{k, p-k, q-k} [7].$$

- iv. The n th Legendre polynomial is defined

$$\text{by } P_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(k!)^2(n-k)!} \left(\frac{x-1}{2}\right)^k. \text{ Thus,}$$

$$P_n(3) = \sum_{k=0}^n \frac{(n+k)!}{(k!)^2(n-k)!}. \text{ By part iii,}$$

$$L_d(n, n) = \sum_{k=0}^n \binom{2n-k}{k, n-k, n-k} = \sum_{k=0}^n \frac{(2n-k)!}{k!(n-k)!^2}. \text{ If } k \text{ is replaced by}$$

$$(n-k), \quad L_d(n, n) = P_n(3). \quad \blacksquare$$

Theorem 2.4: For all $p, q \in \mathbb{N}$,

$$L_d(p, q) = \sum_{k=0}^p \binom{k+q}{k, p-k, k+q-p}$$

where k is the number of east moves.

Proof: If the lattice path from $(0, 0)$ to (p, q) has k east moves, then the lattice path has $p - k$ diagonal moves and $q - (p - k)$ north moves, and hence $k + q$ moves in all. Hence, the set of lattice paths with diagonal moves allowed that has k east moves is in one-to-one correspondence with the set of words that have k E's, $p - k$ D's, and $k + q - p$ N's. Thus, the number of these lattice paths

$$\text{is } \binom{k+q}{k, p-k, k+q-p} \text{ for } k = 0 \dots p. \blacksquare$$

Theorem 2.5: For all $p, q \in \mathbb{N}$,

$$L_d(p, q) = \sum_{k=0}^q \binom{k+p}{k, q-k, k+p-q}$$

where k is the number of north moves.

Proof: If the lattice path from $(0, 0)$ to (p, q) has k north moves, then the lattice path has $q - k$ diagonal moves and $p - (q - k)$ east moves, and hence $k + p$ moves in all. Hence, the set of lattice paths with diagonal moves allowed that has k north moves is in one-to-one correspondence with the set of words that have k N's, $q - k$ D's, and $k + p - q$ E's. The number of these lattice paths is

$$\binom{k+p}{k, q-k, k+p-q} \text{ for } k = 0 \dots q. \blacksquare$$

Theorem 2.6: For all $p, q \in \mathbb{N}$,

$$L_d(p, q) = \sum_{k=0}^{\min\{p, q\}} \binom{p}{k} \binom{q}{k} 2^k$$

where k is the number of east-to-north and diagonal moves.

Proof: To determine the number of lattice paths from $(0, 0)$ to (p, q) with k east-then-north or diagonal moves, we will consider each point (i, j) attained after a diagonal move or after an east-then-north move as a corner on a city block. Such a block will be represented by A_{ij} where $A_{ij} = \{(x, y): i - 1 < x < i, j - 1 < y < j\}$. The k blocks that are traversed by a diagonal or by an east-then-north move must be chosen. If $A_{i_1 j_1}$ and $A_{i_2 j_2}$ are such blocks, then $i_1 \neq i_2$ and $j_1 \neq j_2$. Hence, in

choosing A_{ij} , there are $\binom{p}{k}$ choices for i and $\binom{q}{k}$ choices for j . Because these

blocks must be crossed by either traveling east-then-north or diagonally, there are 2^k ways to cross these blocks. Once the blocks are chosen, the rest of the path is determined, having strings of north moves followed by east moves. For instance, if A_{ij} is the first chosen block, the path from $(0, 0)$ to $(i - 1, j - 1)$ consists of $i - 1$ N's followed by $j - 1$ E's. ■

Theorem 2.7: The generating function for $L_d(p, q)$,

$$\sum_{p \geq 0} \sum_{q \geq 0} L_d(p, q) x^p y^q = \frac{1}{1 - x - y - xy}.$$

Proof 1:
$$\sum_{p \geq 0} \sum_{q \geq 0} L_d(p, q) x^p y^q = \sum_{p \geq 0} \sum_{q \geq 0} \sum_{k=0}^{\min\{p, q\}} \binom{p+q-k}{k, p-k, q-k} x^p y^q$$

$$= \sum_{p \geq 0} \sum_{k \geq 0} \sum_{q \geq k} \binom{q}{k} \binom{p+q-k}{p-k} x^p y^q = \sum_{p \geq 0} \sum_{k \geq 0} \sum_{q \geq 0} \binom{q}{k} \binom{p+q-k}{p-k} x^p y^q$$

$$\begin{aligned}
&= \sum_{k \geq 0} \sum_{q \geq 0} \sum_{p \geq 0} \binom{q}{k} \binom{p+q-k}{p-k} x^p y^q = \sum_{k \geq 0} \sum_{q \geq 0} \binom{q}{k} y^q x^k \sum_{p \geq k} \binom{p+q-k}{p-k} x^{p-k} \\
&= \sum_{k \geq 0} \sum_{q \geq 0} \binom{q}{k} y^q x^k \sum_{m \geq 0} \binom{m+q}{m} x^m = \sum_{k \geq 0} \sum_{q \geq 0} \binom{q}{k} y^q x^k \left(\frac{1}{1-x} \right)^{q+1} \quad \text{Let } m = p - k. \\
&= \frac{1}{1-x} \sum_{k \geq 0} x^k \sum_{q \geq 0} \binom{q}{k} \left(\frac{y}{1-x} \right)^q = \frac{1}{1-x} \sum_{k \geq 0} x^k \frac{\left(\frac{y}{1-x} \right)^k}{\left(1 - \frac{y}{1-x} \right)^{k+1}} \\
&= \frac{1}{1-x-y} \sum_{k \geq 0} \left(\frac{xy}{1-x-y} \right)^k = \frac{1}{1-x-y-xy}. \quad \blacksquare
\end{aligned}$$

Proof 2:

$$\begin{aligned}
\sum_{p,q \geq 0} L_d(p,q) x^p y^q &= \sum_{p,q \geq 0} \sum_{k \leq p,q} \binom{p+q-k}{p-k, q-k, k} x^p y^q \\
&= \sum_{k \geq 0} \sum_{p,q \geq k} \binom{p+q-k}{p-k, q-k, k} x^{p-k} y^{q-k} (xy)^k \\
&= \sum_{n \geq 0} \sum_{k \geq 0} \sum_{\substack{p,q \geq k \\ p+q-k=n}} \binom{n}{p-k, q-k, k} x^{p-k} y^{q-k} (xy)^k
\end{aligned}$$

$$= \sum_{n \geq 0} \sum_{\substack{k, i, j \geq 0 \\ i+j+k=n \\ \text{iff} \\ p+q-k=n}} \binom{n}{i, j, k} x^i y^j (xy)^k = \sum_{n \geq 0} (x + y + xy)^n \quad \text{Let } i = p - k, j = q - k.$$

$$= \frac{1}{1 - x - y - xy}.$$

Proof 3: Let $F(x, y) := \sum_{p, q \geq 0} L_d(p, q) x^p y^q$. Then, by Theorem 2.1,

$$F(x, y) = \sum_{p \geq 0} L_d(p, 0) x^p y^0 + \sum_{q \geq 0} L_d(0, q) x^0 y^q - L_d(0, 0) x^0 y^0$$

$$+ x \sum_{p, q > 0} L_d(p-1, q) x^{p-1} y^q + y \sum_{p, q > 0} L_d(p, q-1) x^p y^{q-1}$$

$$+ xy \sum_{p, q > 0} L_d(p-1, q-1) x^{p-1} y^{q-1}$$

$$= \frac{1}{1-x} + \frac{1}{1-y} - 1 + x \sum_{\substack{p \geq 0, \\ q > 0}} L_d(p, q) x^p y^q$$

$$+ y \sum_{\substack{p > 0, \\ q \geq 0}} L_d(p, q) x^p y^q + xy \sum_{p, q \geq 0} L_d(p, q) x^p y^q$$

$$= \frac{1}{1-x} + \frac{1}{1-y} - 1 + x \left(\sum_{p, q \geq 0} L_d(p, q) x^p y^q - \sum_{p \geq 0} L_d(p, 0) x^p y^0 \right)$$

$$+ y \left(\sum_{p,q \geq 0} L_d(p,q) x^p y^q - \sum_{q \geq 0} L_d(0,q) x^0 y^q \right) + xy \sum_{p,q \geq 0} L_d(p,q) x^p y^q$$

$$= 1 + xF(x,y) + yF(x,y) + xyF(x,y)$$

and so, $F(x,y)(1-x-y-xy)=1$, i.e., $F(x,y) = \frac{1}{1-x-y-xy}$. ■

2.2. Paths that Do Not Rise Above the Diagonal: $L_d^*(p, q)$.

The number of lattice paths with diagonal moves allowed that may touch but not cross the diagonal $y = x$ will be denoted $L_d^*(p, q)$. A recursive formula for $L_d^*(p, q)$ is the following.

Theorem 2.8: For $p \in \mathbb{N}$, $L_d^*(p, 0) = 1$. If $p < q$, $L_d^*(p, q) = 0$. For $p, q \in \mathbb{P}$ with $p \geq q$,

$$L_d^*(p, q) = L_d^*(p, q-1) + L_d^*(p-1, q) + L_d^*(p-1, q-1). \quad (2.2)$$

Proof: From the point $(0, 0)$ to $(p, 0)$, the only path is the one that consists of p E's.

Hence, $L_d^*(p, 0) = 1$. If $p < q$, the lattice path must cross the diagonal to touch the point (p, q) . Since it is not allowed to cross the diagonal, $L_d^*(p, q) = 0$.

Suppose $p, q \in \mathbb{P}$ with $p \geq q$. Then, the LHS of 2.2 counts the number of lattice paths that do not cross the diagonal $y = x$ by definition. The RHS count this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q) is $(p, q-1)$,
- ii. The category in which the last point visited before (p, q) is $(p-1, q)$. If $p-1 < q$, then this category would be empty, and
- iii. The category in which the last point visited before (p, q) is $(p-1, q-1)$. ■

Several values $L_d^*(p, q)$ are given in Table 2.2.

$L_d^*(p, q)$

$p \backslash q$	0	1	2	3	4
0	1	0	0	0	0
1	1	2	0	0	0
2	1	4	6	0	0
3	1	6	16	22	0
4	1	8	30	68	90

Table 2.2

Theorem 2.9: For all $p, q \in \mathbb{N}$,

$$L_d^*(p, q) = \begin{cases} 0 & \text{if } p < q \\ \sum_{k=0}^q \frac{p-q+1}{p+q-k+1} \binom{p+q-k+1}{k, p-k+1, q-k} & \text{if } p \geq q \end{cases}$$

Proof: If $p < q$, $L_d^*(p, q) = 0$ by Theorem 2.8.

Suppose $p \geq q$. A lattice path from $(0, 0)$ to (p, q) has k diagonal moves for some $k = 0, \dots, q$. The number of lattice paths with k diagonal moves is counted as follows. Because there are k diagonal moves, there are $p - k$ east moves and $q - k$ north moves. Thus, there are a total of $p + q - k$ moves. Thus, there are $p + q - k$ positions in the word representation of such a path. If we choose k of these positions to be D's, they can be placed in $\binom{p+q-k}{k}$ ways. Now, there are

$p + q - 2k$ positions left. We want to arrange the $p - k$ E's and $q - k$ N's so that at any point in the word there are at least as many E's and N's. By Theorem 1.7, this

can be done in $L^*(p-k, q-k) = \frac{p-q+1}{p-k+1} \binom{p+q-2k}{p-k}$ ways, so

$$L_d^*(p, q) = \sum_{k=0}^q \binom{p+q-k}{k} \frac{p-q+1}{p-k+1} \binom{p+q-2k}{p-k}$$

$$= \sum_{k=0}^q \frac{p-q+1}{p-k+1} \binom{p+q-k}{k, p-k, q-k}. \blacksquare$$

Theorem 2.9, which is due to Wagner [8], has not appeared in the literature. The only result of this sort to have appeared in the literature is the following easy corollary, due to Moser and Zayachkowski [6].

Corollary 2.9.1: For $p \in \mathbf{P}$,

$$L_d^*(p, p) = 1 + \sum_{k=0}^{p-1} \frac{1}{p-k} \binom{2p-k}{k, p-k+1, p-k-1}.$$

2.3. Paths Strictly Below the Diagonal Except at $(0, 0)$: $L_d^{**}(p, q)$

Now, let us consider the lattice paths from $(0, 0)$ to (p, q) that do not cross or touch the diagonal except at the point $(0, 0)$. The number of these paths is denoted $L_d^{**}(p, q)$. The following is a recursive formula for $L_d^{**}(p, q)$.

Theorem 2.10: For $p \in \mathbf{N}$, $L_d^{**}(p, 0) = 1$. For $q \in \mathbf{P}$, $L_d^{**}(0, q) = 0$. For $p, q \in \mathbf{P}$, with $p \leq q$, $L_d^{**}(p, q) = 0$. For $p, q \in \mathbf{P}$ with $p > q$,

$$L_d^{**}(p, q) = L_d^{**}(p, q-1) + L_d^{**}(p-1, q) + L_d^{**}(p-1, q-1). \quad (2.3)$$

Proof: From the point $(0, 0)$ to the point $(p, 0)$, the only path is the one that consists of p E's. Hence, $L_d^{**}(p, 0) = 1$. There is not a path from $(0, 0)$ to $(0, q)$ because the lattice path would have to cross the line $y = x$, so $L_d^{**}(0, q) = 0$. If $p \leq q$, the lattice path must cross or touch the diagonal $y = x$ to reach the point (p, q) which the lattice path is not allowed to do. Hence, $L_d^{**}(p, q) = 0$.

Suppose $p, q \in \mathbf{P}$ with $p > q$. Then, the LHS of 2.3 counts the number of lattice paths that do not cross or touch the diagonal except at $(0, 0)$ by definition. The RHS counts this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q) is $(p, q-1)$,
- ii. The category in which the last point visited before (p, q) is $(p-1, q)$. If $p-1 = q$, this category is empty, so $L_d^{**}(p-1, q) = 0$, and

iii. The category in which the last point visited before (p, q) is $(p - 1, q - 1)$. ■

Several values of $L_d^{**}(p, q)$ are listed in Table 2.3.

$L_d^{**}(p, q)$

$p \backslash q$	0	1	2	3	4
0	1	0	0	0	0
1	1	0	0	0	0
2	1	2	0	0	0
3	1	4	6	0	0
4	1	6	16	22	0

Table 2.3

As in the previous cases, we would like to have a closed form expression for $L_d^{**}(p, q)$.

Theorem 2.11: For $p > q \geq 0$,

$$L_d^{**}(p, q) = L_d^*(p - 1, q)$$

and so, a closed form expression for $L_d^{**}(p, q)$ is :

$$L_d^{**}(p, q) = \sum_{k=0}^q \frac{p - q}{p + q - k} \binom{p + q - k}{k, p - k, q - k}.$$

Proof: The first move for any lattice path from $(0, 0)$ to (p, q) that does not touch $y = x$ except at $(0, 0)$ is to the point $(1, 0)$. The number of such paths from $(1, 0)$ to (p, q) is the same as the number of lattice paths from $(0, 0)$ to $(p - 1, q)$ that may touch but not cross the diagonal $y = x$. Thus, by Theorem 2.9,

$$L_d^{**}(p, q) = L_d^*(p - 1, q) = \sum_{k=0}^q \frac{p - q}{p + q - k} \binom{p + q - k}{k, p - k, q - k}. \quad \blacksquare$$

2.4. Paths Strictly Below the Diagonal

Except at $(0, 0)$ and at (p, q) if $p = q$: $L_d^{**}(p, q)$

We now want to consider the lattice paths that do not touch the diagonal except at the points $(0, 0)$ and (p, q) if $p = q$. The number of such lattice paths will be denoted

$L_d^{(**)}(p, q)$. Again, we would like to find a recursive formula for $L_d^{(**)}(p, q)$. However as with $L^{(**)}(p, q)$, the values of $L_d^{(**)}(p, p - 1)$ prevent finding a simple recursive formula. Several values of $L_d^{(**)}(p, q)$ are listed in Table 2.4.

$L_d^{(**)}(p, q)$	$p \backslash q$	0	1	2	3	4
	0	1	0	0	0	0
	1	1	1	0	0	0
	2	1	2	2	0	0
	3	1	4	6	6	0
	4	1	6	16	22	22

Table 2.4

Of course, the usual three term recurrence holds for $L_d^{(**)}(p, q)$ if $p - 2 \geq q \geq 1$, but the boundary conditions that accompany this recurrence are complicated to state. Fortunately, $L_d^{(**)}(p, q)$ may be specified in terms of $L_d^{**}(p, q)$ and $L_d^*(p, q)$, as indicated by the following theorems.

Theorem 2.12: For $q \in \mathbb{N}$, and $p > q$,

$$L_d^{(**)}(p, q) = L_d^{**}(p, q).$$

Proof: If $p > q$, the lattice path can never touch the point (p, q) because $p \neq q$. Thus,

$L_d^{(**)}(p, q)$ and $L_d^{**}(p, q)$ both count the number of lattice paths with diagonal moves allowed that never touch the line $y = x$ except at the point $(0, 0)$. ■

Note that $L_d^{(**)}(p, p) \neq L_d^{**}(p, p)$, because $L_d^{(**)}(p, p)$ counts the number of lattice paths from $(0, 0)$ to (p, p) that stay strictly below $y = x$ but are allowed to touch (p, p) , $L_d^{(**)}(p, p) > 0$. Instead, we have the following:

Theorem 2.13: For $p \in \mathbb{N}$,

$$L_d^*(p, p) = L_d^{(**)}(p + 1, p + 1).$$

Proof: $L_d^{(**)}(p + 1, p + 1)$ counts the number of lattice paths from $(0, 0)$ to $(p + 1, p + 1)$ with diagonal moves allowed that may not touch $y = x$ except at $(0, 0)$ and $(p + 1, p + 1)$. Thus, the first move from $(0, 0)$ to $(1, 0)$ and the last move from

$(p + 1, p)$ to $(p + 1, p + 1)$ have already been determined. Thus, the number of lattice paths from $(1, 0)$ to $(p + 1, p)$ is the same as the number of lattice paths from $(0, 0)$ to (p, p) with diagonal moves allowed that may touch but not cross $y = x$, $L_d^*(p, p)$. ■

2.5 Notes

Theorem 2.1 is found in [5]. The source of Theorem 2.2 is Wagner [8]. Theorem 2.3 iii and iv and their proofs are found in [6], with the definition of a trinomial coefficient in [7]. The source of Theorem 2.6 is Davis [1]. The source of Theorem 2.7 and 2.9 is Dr. Wagner [8]. Theorem 2.8 is found in [4]. Corollary 2.9.1 is found in [6]. The source of Theorems 2.11 - 2.13 is Wagner [8].

Chapter 3 Lattice Paths in $\mathbb{R}^3$

3.1. Unrestricted Paths: $L(p, q, r)$

A three-dimensional lattice path without diagonal moves has three possible steps: parallel to the x-axis, parallel to the y-axis, and parallel to the z-axis. The number of these lattice paths from $(0, 0, 0)$ to (p, q, r) is denoted $L(p, q, r)$. The following is a recursive formula for $L(p, q, r)$.

Theorem 3.1: For all $p, q, r \in \mathbb{N}$, $L(p, q, 0) = L(p, q)$, $L(p, 0, r) = L(p, r)$, and $L(0, q, r) = L(q, r)$. For all $p, q, r \in \mathbb{P}$,

$$L(p, q, r) = L(p-1, q, r) + L(p, q-1, r) + L(p, q, r-1). \quad (3.1)$$

Proof: Any path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy-plane. Thus, the lattice path is a path in $\mathbb{R}^2$. The number of such paths is $L(p, q)$. Similarly, $L(p, 0, r) = L(p, r)$ and $L(0, q, r) = L(q, r)$.

The LHS of 3.1 counts the set of three-dimensional lattice paths by definition. The RHS of 3.1 counts this set in three disjoint exhaustive categories:

- i. The category of paths in which the last point visited before (p, q, r) is $(p-1, q, r)$,
- ii. The category of paths in which the last point visited before (p, q, r) is $(p, q-1, r)$, and
- iii. The category of paths in which the last point visited before (p, q, r) is $(p, q, r-1)$. ■

We now want a closed form expression for $L(p, q, r)$.

Theorem 3.2: For $p, q, r \in \mathbb{N}$,

$$L(p, q, r) = \binom{p+q+r}{p, q, r}.$$

Proof: Any lattice path from $(0, 0, 0)$ to (p, q, r) consists of p moves parallel to the x-axis, q moves parallel to the y-axis, and r moves parallel to the z-axis. The set of lattice paths from $(0, 0, 0)$ to (p, q, r) is in one-to-one correspondence with the set

of words of length $p + q + r$ consisting of p X's, q Y's, and r Z's. First, we choose the p positions in which to place the X's, then we choose the q positions to place the Y's. Finally, we choose positions for the r Z's. The number of sequences of

p X's, q Y's, and r Z's is $\binom{p+q+r}{p, q, r}$ as proven in [7]. ■

Theorem 3.3: The generating function for $L(p, q, r)$,

$$\sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} L(p, q, r) x^p y^q z^r = \frac{1}{1-x-y-z}.$$

Proof 1:

$$\begin{aligned} \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} L(p, q, r) x^p y^q z^r &= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \binom{p+q+r}{p, q, r} x^p y^q z^r \\ &= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \binom{p+q+r}{p} \binom{q+r}{r} x^p y^q z^r = \sum_{q \geq 0} y^q \sum_{r \geq 0} \binom{q+r}{r} z^r \sum_{p \geq 0} \binom{p+q+r}{p} x^p \\ &= \sum_{q \geq 0} y^q \sum_{r \geq 0} \binom{q+r}{r} z^r \frac{1}{(1-x)^{q+r+1}} = \frac{1}{1-x} \sum_{q \geq 0} \left(\frac{y}{1-x} \right)^q \sum_{r \geq 0} \binom{q+r}{r} \left(\frac{z}{1-x} \right)^r \\ &= \frac{1}{1-x} \sum_{q \geq 0} \left(\frac{y}{1-x} \right)^q \left(\frac{1-x}{1-x-z} \right)^{q+1} = \frac{1}{1-x-z} \sum_{q \geq 0} \left(\frac{y}{1-x-z} \right)^q \\ &= \frac{1}{1-x-z} \frac{1-x-z}{1-x-y-z} = \frac{1}{1-x-y-z}. \end{aligned}$$

Proof 2:

$$\sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} L(p, q, r) x^p y^q z^r = \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \binom{p+q+r}{p, q, r} x^p y^q z^r$$

$$= \sum_{n \geq 0} \sum_{p+q+r=n} \binom{n}{p, q, r} x^p y^q z^r = \sum_{n \geq 0} (x + y + z)^n = \frac{1}{1 - x - y - z}. \quad \blacksquare$$

3.2. Restricted Lattice Paths: $L^*(p, q, r)$, $L^{**}(p, q, r)$, and $L^*(p, q, r)$

Lattice paths in $\mathbb{R}^3$ may be restricted in many different ways. We will consider three of these ways. First, we will consider the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path, $i \geq j$. The number of such paths will be denoted $L^*(p, q, r)$. The following is a recursive formula for $L^*(p, q, r)$.

Theorem 3.4: For all $p, q, r \in \mathbb{N}$, $L^*(p, 0, r) = L(p, r)$, $L^*(p, q, 0) = L^*(p, q)$, and if $p < q$ then $L^*(p, q, r) = 0$. If $p, q, r \in \mathbb{P}$ and $p \geq q$,

$$L^*(p, q, r) = L^*(p-1, q, r) + L^*(p, q-1, r) + L^*(p, q, r-1). \quad (3.2)$$

Proof: Suppose $p < q$. Then at the lattice point (p, q, r) , p is less than q . Since $i \geq j$ for each lattice point, $L^*(p, q, r) = 0$.

A lattice path from $(0, 0, 0)$ to $(p, 0, r)$ remains on the xz -plane. Since every lattice point $(i, 0, k)$ $i \geq j$ satisfies $i \geq 0$, the moves parallel to the z -axis are unrestricted. Thus, the set of lattice paths from $(0, 0, 0)$ to $(p, 0, r)$ such that for each lattice point (i, j, k) $i \geq j$ is in one-to-one correspondence with the set of lattice paths from $(0, 0)$ to (p, r) . Thus, $L^*(p, 0, r) = L(p, r)$.

A lattice path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane. Thus, the set of lattice paths from $(0, 0, 0)$ to $(p, q, 0)$ such that for each lattice point $(i, j, 0)$ $i \geq j$ is in one-to-one correspondence with the set of lattice paths from $(0, 0)$ to (p, q) that may touch but not cross the line $y = x$. Thus, $L^*(p, q, 0) = L^*(p, q)$.

Suppose $p \geq q \geq 1$. Then, by definition the LHS of 3.2 counts the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path $i \geq j$. The RHS counts this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p-1, q, r)$.
If $p = q$, this category is empty, and $L^*(p-1, q, r) = 0$,
- ii. The category in which the last point visited before (p, q, r) is $(p, q-1, r)$,
and

- iii. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$.

■

We now want a closed form expression for $L^*(p, q, r)$.

Theorem 3.5: For all $p, q, r \in \mathbb{N}$,

$$\begin{aligned} L^*(p, q, r) &= L^*(p, q) \binom{p+q+r}{r} \\ &= \frac{p-q+1}{p+1} \binom{p+q+r}{p, q, r}. \end{aligned}$$

Proof: Any lattice path from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) $i \geq j$ has p moves parallel to the x -axis, q moves parallel to the y -axis, and r moves parallel to the z -axis. Thus, the corresponding word representation has p X's, q

Y's, and r Z's. There are $\binom{p+q+r}{r}$ ways to place the r Z's. Next, we want to

arrange the p X's and q Y's so that at any point in the word representation there are at least as many X's as there are Y's. By Theorem 1.7, there are

$$L^*(p, q) = \frac{p-q+1}{p+1} \binom{p+q}{q} \text{ ways to place the X's and Y's. Thus,}$$

$$\begin{aligned} L^*(p, q, r) &= L^*(p, q) \binom{p+q+r}{r} \\ &= \frac{p-q+1}{p+1} \binom{p+q}{q} \binom{p+q+r}{r} = \frac{p-q+1}{p+1} \binom{p+q+r}{p, q, r}. \quad \blacksquare \end{aligned}$$

We now want to apply this to the ballot problem. Suppose that in an election,

there are three candidates. Candidate A receives p votes, Candidate B receives q votes, and Candidate C receives r votes with $p \geq q$. We want to find the probability that if the votes are recorded randomly Candidate A never trails Candidate B at any point in the election.

Theorem 3.6: For $p, q, r \in \mathbb{N}$ and $p \geq q$, if Candidate A receives p votes, Candidate B receives q votes, and Candidate C receives r votes, then the probability

$$\text{that Candidate A never trails Candidate B is } \frac{p - q + 1}{p + 1}.$$

Proof: If we record each vote as it is cast, we will have a word of p A's, q B's, and r C's.

The set of these words is in one-to-one correspondence to the set of lattice paths

from $(0, 0, 0)$ to (p, q, r) . By Theorem 3.2, there are $\binom{p + q + r}{p, q, r}$ paths from

$(0, 0, 0)$ to (p, q, r) . The number of lattice paths from $(0, 0, 0)$ to (p, q, r) such that

for each lattice point (i, j, k) $i \geq j$ is $\frac{p - q + 1}{p + 1} \binom{p + q + r}{p, q, r}$ by Theorem 3.5.

Thus, the probability that Candidate A never trails Candidate B is $\frac{p - q + 1}{p + 1}$. ■

This probability does not depend on r and is identical with the probability calculated in Section 1.2 for the two-person election, which is intuitively reasonable.

Now we want to consider the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path $i > j$. The number of such lattice paths will be denoted $L^{**}(p, q, r)$. The following is a recursive formula for $L^{**}(p, q, r)$.

Theorem 3.7: For all $p, q, r \in \mathbb{N}$, $L^{**}(p, q, 0) = L^{**}(p, q)$, $L^{**}(p, 0, r) = L(p, r)$, and if

$p \leq q$, $L^{**}(p, q, r) = 0$. If $p, q, r \in \mathbb{P}$ and $p > q$,

$$L^{**}(p, q, r) = L^{**}(p - 1, q, r) + L^{**}(p, q - 1, r) + L^{**}(p, q, r - 1). \quad (3.3)$$

Proof: Suppose $p \leq q$. Then at the lattice point (p, q, r) $p \leq q$. Since $i > j$ for each lattice

point, $L^{**}(p, q, r) = 0$.

A lattice path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane. Thus, the set of lattice paths from $(0, 0, 0)$ to $(p, q, 0)$ such that for each point $(i, j, 0)$ $i > j$ is in one-to-one correspondence with the set of lattice paths from $(0, 0)$ to (p, q) that may not touch the line $y = x$. Thus, $L^{**}(p, q, 0) = L^{**}(p, q)$.

A lattice path from $(0, 0, 0)$ to $(p, 0, r)$ remains on the xz -plane. Since every lattice point $(i, 0, k)$ satisfies $i \geq 0$, the moves parallel to the z -axis are unrestricted. Thus, the set of lattice paths from $(0, 0, 0)$ to $(p, 0, r)$ such that for each lattice point (i, j, k) $i > j$ is in one-to-one correspondence with the set of lattice paths from $(0, 0)$ to (p, r) . Thus, $L^{**}(p, 0, r) = L(p, r)$.

Suppose $p > q$. Then, the LHS of 3.3 counts the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path $i > j$ by definition. The RHS of 3.3 counts this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p - 1, q, r)$.
If $p - 1 = q$, this category is empty, and $L^{**}(p - 1, q, r) = 0$.
- ii. The category in which the last point visited before (p, q, r) is $(p, q - 1, r)$,
and
- iii. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$. ■

We can, also, find the following closed form expression for $L^{**}(p, q, r)$.

Theorem 3.8: For all $p, q, r \in \mathbb{N}$,

$$L^{**}(p, q, r) = L^{**}(p, q) \binom{p + q + r}{r} = \frac{p - q}{p + q} \binom{p + q + r}{p, q, r}.$$

Proof: Any lattice path from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) $i > j$ has p moves parallel to the x -axis, q -moves parallel to the y -axis, and r moves parallel to the z -axis. Thus, the corresponding word representation has p X 's, q

Y 's, and r Z 's. There are $\binom{p + q + r}{r}$ ways to place the r Z 's. Next, we want to

arrange the p X's and q Y's so that at any point in the word representation there

are more X's than Y's. By Theorem 1.11, there are $L^{**}(p, q) = \frac{p-q}{p+q} \binom{p+q}{q}$

ways to place the X's and Y's. Thus,

$$\begin{aligned} L^{**}(p, q, r) &= L^{**}(p, q) \binom{p+q+r}{r} = \frac{p-q}{p+q} \binom{p+q}{q} \binom{p+q+r}{r} \\ &= \frac{p-q}{p+q} \binom{p+q+r}{p, q, r}. \quad \blacksquare \end{aligned}$$

As in previous cases, this can be applied to the ballot problem. Suppose we have an election with three candidates in which Candidate A receives p votes, Candidate B receives q votes, and Candidate C receives r votes, where $p > q$. We want to determine the probability that if the votes are recorded randomly that Candidate A is always strictly ahead of Candidate B in each partial tally of the votes.

Theorem 3.9: For $p, q, r \in \mathbb{N}$ and $p > q$, if Candidate A receives p votes, Candidate B receives q votes, and Candidate C receives r votes, then the probability

that Candidate A always leads Candidate B is $\frac{p-q}{p+q}$.

Proof: If we record each vote as it is cast, we will have a word consisting of p A's, q B's, and r C's. The set of these words is in one-to-one correspondence to the set lattice

paths from $(0, 0, 0)$ to (p, q, r) . By Theorem 3.2, there are $\binom{p+q+r}{p, q, r}$ paths

from $(0, 0, 0)$ to (p, q, r) . The number of lattice paths from $(0, 0, 0)$ to (p, q, r)

such that for each lattice point (i, j, k) $i > j$ is $\frac{p-q}{p+q} \binom{p+q+r}{p, q, r}$ by Theorem

3.8. Thus, the probability that Candidate A always leads Candidate B is $\frac{p-q}{p+q}$.

■

The third type of restricted lattice path that we will consider is the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path $i \geq j \geq k$. The number of these lattice paths will be denoted $L^*(p, q, r)$. The following is a recursive formula for $L^*(p, q, r)$.

Theorem 3.10: For all $p, q, r \in \mathbb{N}$, $L^*(p, q, r) = 0$ if $p < q$ or $q < r$. Also, $L^*(p, q, 0) = L^*(p, q)$. If $p \geq q \geq r$,

$$L^*(p, q, r) = L^*(p-1, q, r) + L^*(p, q-1, r) + L^*(p, q, r-1). \quad (3.4)$$

Proof: Suppose $p < q$ or $q < r$. Then, a lattice path from $(0, 0, 0)$ to (p, q, r) has at least one lattice point, (p, q, r) , for instance, such that $i < j$ or $j < k$, so $L^*(p, q, r) = 0$.

A lattice path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane. The number of lattice paths from $(0, 0, 0)$ to $(p, q, 0)$ such that for each lattice point $(i, j, 0)$ $i \geq j$ is in one-to-one correspondence with the number of lattice paths from $(0, 0)$ to (p, q) which may touch but not cross the line $y = x$. Thus, $L^*(p, q, 0) = L^*(p, q)$.

Suppose $p \geq q \geq r \geq 1$. Then, the LHS of 3.4 counts the set of lattice paths from $(0, 0, 0)$ to (p, q, r) such that for each lattice point (i, j, k) on the path $i \geq j \geq k$ by definition. The RHS of 3.4 counts this set in three disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p-1, q, r)$.
If $p = q$, then the category is empty, and $L^*(p-1, q, r) = 0$.
- ii. The category in which the last point visited before (p, q, r) is $(p, q-1, r)$.
If $q = r$, then the category is empty, and $L^*(p, q-1, r) = 0$, and

iii. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$.

■

We now present a closed form of $L^*(p, q, r)$.

Theorem 3.11: For $p \geq q \geq r \geq 0$,

$$L^*(p, q, r) = (p + q + r)! \det \begin{bmatrix} \frac{1}{p!} & \frac{1}{(p+1)!} & \frac{1}{(p+2)!} \\ \frac{1}{(q-1)!} & \frac{1}{q!} & \frac{1}{(q+1)!} \\ \frac{1}{(r-2)!} & \frac{1}{(r-1)!} & \frac{1}{r!} \end{bmatrix},$$

where $\frac{1}{a!} := 0$ if $a < 0$.

This formula was first conjectured by Grossman [3] and proved by Zeilberger [9], who actually established an n -dimensional version of this result. We omit the proof.

Setting $r = 0$ in Theorem 3.11 gives another proof of Theorem 1.7:

$$\begin{aligned} L^*(p, q) &= (p + q)! \det \begin{bmatrix} \frac{1}{p!} & \frac{1}{(p+1)!} \\ \frac{1}{(q-1)!} & \frac{1}{q!} \end{bmatrix} \\ &= (p + q)! \left(\frac{1}{p!q!} - \frac{1}{(p+1)!(q-1)!} \right) = (p + q)! \left(\frac{p+1-q}{(p+1)!q!} \right) \\ &= \frac{p-q+1}{p+1} \frac{(p+q)!}{p!q!} = \frac{p-q+1}{p+1} \binom{p+q}{q}. \end{aligned}$$

Again, this type of lattice path can be applied to the ballot problem. This time Candidate A receives p votes, Candidates B receives q votes, and Candidate C receives r votes such that $p \geq q \geq r$. We want to find the probability that if the votes are recorded randomly Candidate A never trails Candidate B and Candidate B never trails Candidate C.

Theorem 3.12: For $p \geq q \geq r \geq 0$, the probability that Candidate A never trails Candidate

B and Candidate B never trails Candidate C is

$$1 + \frac{(r-1)r}{(p+1)(q+1)} + \frac{r(q-r+1)}{(p+2)(p+1)} - \frac{q}{p+1} - \frac{r}{q+1}.$$

Proof: If we record each vote as it is cast, we will have a word of p A's, q B's, and r C's.

The set of these words is in one-to-one correspondence to the set of lattice paths from $(0, 0, 0)$ to (p, q, r) . The number of lattice paths such that for each lattice point (i, j, k) $i \geq j \geq k$ is $L^*(p, q, r)$. By Theorem 3.1, the total number of lattice

paths is $\binom{p+q+r}{p, q, r}$. Hence, the probability that Candidate A never trail

Candidate B who never trails Candidate C is

$$(p+q+r)! \det \begin{bmatrix} \frac{1}{p!} & \frac{1}{(p+1)!} & \frac{1}{(p+2)!} \\ \frac{1}{(q-1)!} & \frac{1}{q!} & \frac{1}{(q+1)!} \\ \frac{1}{(r-2)!} & \frac{1}{(r-1)!} & \frac{1}{r!} \end{bmatrix} \bigg/ \binom{p+q+r}{p, q, r}$$

which easily simplifies to

$$1 + \frac{(r-1)r}{(p+1)(q+1)} + \frac{r(q-r+1)}{(p+2)(p+1)} - \frac{q}{p+1} - \frac{r}{q+1} \text{ by expanding all the}$$

expressions involved. ■

3.3 Notes

Theorem 3.2 is found in [3]. Theorem 3.5 is found in [5]. The proof of Theorem 3.6 is found in [3]. Theorems 3.8 and 3.9 are in [5]. Theorems 3.11 and 3.12 are in [3].

Chapter 4

Lattice Paths with Diagonal Moves in $\mathbb{R}^3$

4.1. Unrestricted paths: $L_d(p, q, r)$ and $L_p(p, q, r)$

In $\mathbb{R}^3$, there are four types of diagonal moves. A move may be diagonal with respect to the xy -plane, the xz -plane, the yz -plane, or across the cube. A cube diagonal move is from the point (x, y, z) to $(x + 1, y + 1, z + 1)$. These different diagonal moves will be denoted as d_{xy} for xy -diagonal moves, d_{xz} for xz -diagonal moves, d_{yz} for yz -diagonal moves, and d_{xyz} for cube diagonal moves. We will consider lattice paths that allow only cube diagonal moves and lattice paths that allow each of the four types of diagonal moves. First, we will examine those lattice paths that allow only cube diagonal moves. The number of these paths from $(0, 0, 0)$ to (p, q, r) will be denoted $L_d(p, q, r)$. A recursive formula for $L_d(p, q, r)$ is the following:

Theorem 4.1: For $p, q, r \in \mathbb{N}$, $L_d(p, 0, 0) = L_d(0, q, 0) = L_d(0, 0, r) = 1$. For $p, q, r \in \mathbb{P}$,

$$L_d(p, q, 0) = L(p, q), L_d(p, 0, r) = L(p, r), \text{ and } L_d(0, q, r) = L(q, r). \text{ For } p, q, r \in \mathbb{P},$$

$$L_d(p, q, r) = L_d(p, q, r - 1) + L_d(p, q - 1, r) + L_d(p - 1, q, r) + L_d(p - 1, q - 1, r - 1) \quad (4.1)$$

Proof: From the point $(0, 0, 0)$ to $(p, 0, 0)$, the only path is the path that consists of p X's.

Hence, $L_d(p, 0, 0) = 1$. Similarly, $L_d(0, q, 0) = 1$ and $L_d(0, 0, p) = 1$.

Any lattice path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane.

Because a cube diagonal move would cause the lattice path to leave the xy -plane, the lattice path has no diagonal moves. Hence, the number of lattice paths from $(0, 0, 0)$ to $(p, q, 0)$ with cube diagonal moves allowed is the same as the number of lattice paths from $(0, 0)$ to (p, q) . Thus, $L_d(p, q, 0) = L(p, q)$. Similarly, $L_d(p, 0, r) = L(p, r)$ and $L_d(0, q, r) = L(q, r)$.

The LHS of 4.1 counts the set of lattice paths with cube diagonal moves allowed by definition. The RHS counts this set in four disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$,
- ii. The category in which the last point visited before (p, q, r) is $(p, q - 1, r)$,

- iii. The category in which the last point visited before (p, q, r) is $(p - 1, q, r)$,
and
- iv. The category in which the last point visited before (p, q, r) is
 $(p - 1, q - 1, r - 1)$. ■

There are several closed form expressions for $L_d(p, q, r)$.

Theorem 4.2: For all $p, q, r \in \mathbb{N}$,

$$L_d(p, q, r) = \sum_{k=0}^{\min\{p, q, r\}} \binom{p + q + r - 2k}{p - k, q - k, r - k, k}$$

where k is the number of cube diagonal moves.

Proof: Suppose there are k cube diagonal moves where $0 \leq k \leq \min\{p, q, r\}$. Then, there are $p - k$ moves parallel to the x -axis, $q - k$ moves parallel to the y -axis, and $r - k$ moves parallel to the z -axis. The set of lattice paths of this type is clearly in one-to-one correspondence with the set of words with $p - k$ X's, $q - k$ Y's, $r - k$ Z's, and k D's. Thus, the number of lattice paths with k cube diagonal moves is

$$\binom{p + q + r - 2k}{p - k, q - k, r - k, k}. \quad \blacksquare$$

Theorem 4.3: For all $p, q, r \in \mathbb{N}$,

$$L_d(p, q, r) = \sum_{k=0}^p \binom{q + r - p + 2k}{k, k + q - p, k + r - p, p - k}$$

where k is the number of moves parallel to the x -axis.

Proof: Suppose a lattice path with cube diagonal moves allowed has k moves parallel to the x -axis. Then, the lattice path has $p - k$ cube diagonal moves, $k + q - p$ moves parallel to the y -axis, and $k + r - p$ moves parallel to the z -axis. The set of lattice paths of this type is clearly in one-to-one correspondence with the set of words that have k X's, $k + q - p$ Y's, $k + r - p$ Z's, and $p - k$ D's. The number of these

lattice paths is $\binom{q+r-p+2k}{k, k+q-p, k+r-p, p-k}$ for $k = 0, \dots, p$. ■

Theorem 4.4: For all $p, q, r \in \mathbb{N}$,

$$L_d(p, q, r) = \sum_{k=0}^q \binom{p+r-q+2k}{k+p-q, k, k+r-q, q-k}$$

where k is the number of moves parallel to the y -axis.

Proof: Suppose a lattice path with cube diagonal moves allowed has k moves parallel to the y -axis. Then, the lattice path has $q - k$ cube diagonal moves, $k + p - q$ moves parallel to the x -axis, and $k + r - q$ moves parallel to the z -axis. The set of lattice paths of this type is clearly in one-to-one correspondence with the set of words that have $k + p - q$ X 's, k Y 's, $k + r - q$ Z 's, and $q - k$ D 's. The number of these

lattice paths is $\binom{p+r-q+2k}{k+p-q, k, k+r-q, q-k}$ for $k = 0, \dots, q$. ■

Theorem 4.5: For all $p, q, r \in \mathbb{N}$,

$$L_d(p, q, r) = \sum_{k=0}^r \binom{p+q-r+2k}{k+p-r, k+q-r, k, r-k}$$

where k is the number of moves parallel to the z -axis.

Proof: Suppose a lattice path with cube diagonal moves allowed has k moves parallel to the z -axis. Then, the lattice path has $r - k$ cube diagonal moves, $k + p - r$ moves parallel to the x -axis, and $k + q - r$ moves parallel to the y -axis. The set of lattice paths of this type is clearly in one-to-one correspondence with the set of words that have $k + p - r$ X 's, $k + q - r$ Y 's, k Z 's, and $r - k$ D 's. The number of these

lattice paths is $\binom{p+q-r+2k}{k+p-r, k+q-r, k, r-k}$ for $k = 0, \dots, r$. ■

Theorem 4.6: The generating function for $L_d(p, q, r)$,

$$\sum_{p,q,r \geq 0} L_d(p, q, r) x^p y^q z^r = \frac{1}{1 - x - y - z - xyz}.$$

Proof 1:

$$\begin{aligned} \sum_{p,q,r \geq 0} L_d(p, q, r) x^p y^q z^r &= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \sum_{k=0}^{\min\{p,q,r\}} \binom{p+q+r-2k}{p-k, q-k, r-k, k} x^p y^q z^r \\ &= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{k \geq 0} \sum_{r \geq k} \binom{p+q+r-2k}{p-k, q-k, r-k, k} x^p y^q z^r \\ &= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{k \geq 0} \sum_{r \geq k} \binom{r}{k} \binom{q+r-k}{q-k} \binom{p+q+r-2k}{p-k} x^p y^q z^r \\ &= \sum_{k \geq 0} \sum_{r \geq 0} \binom{r}{k} z^r \sum_{q \geq 0} \binom{q+r-k}{q-k} y^q \sum_{p \geq 0} \binom{p+q+r-2k}{p-k} x^p \\ &= \sum_{k \geq 0} x^k \sum_{r \geq 0} \binom{r}{k} z^r \sum_{q \geq 0} \binom{q+r-k}{q-k} y^q \sum_{p \geq k} \binom{p+q+r-2k}{p-k} x^{p-k} \\ &= \sum_{k \geq 0} x^k \sum_{r \geq 0} \binom{r}{k} z^r \sum_{q \geq 0} \binom{q+r-k}{q-k} y^q \sum_{m \geq 0} \binom{m+q+r-k}{m} x^m \text{ Let } m = p - k \\ &= \sum_{k \geq 0} x^k \sum_{r \geq 0} \binom{r}{k} z^r \sum_{q \geq 0} \binom{q+r-k}{q-k} y^q \left(\frac{1}{1-x} \right)^{q+r-k+1} \end{aligned}$$

$$= \frac{1}{1-x} \sum_{k \geq 0} x^k (1-x)^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x} \right)^r \sum_{q \geq 0} \binom{q+r-k}{q-k} \left(\frac{y}{1-x} \right)^q$$

$$= \frac{1}{1-x} \sum_{k \geq 0} x^k y^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x} \right)^r \sum_{q \geq k} \binom{q+r-k}{q-k} \left(\frac{y}{1-x} \right)^{q-k}$$

$$= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x} \right)^r \sum_{t \geq 0} \binom{t+r}{t} \left(\frac{y}{1-x} \right)^t \quad \text{Let } t = q - k$$

$$= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x} \right)^r \left(\frac{1}{1 - \frac{y}{1-x}} \right)^{r+1}$$

$$= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x} \right)^r \left(\frac{1-x}{1-x-y} \right)^{r+1}$$

$$= \frac{1}{1-x-y} \sum_{k \geq 0} (xy)^k \sum_{r \geq 0} \binom{r}{k} \left(\frac{z}{1-x-y} \right)^r$$

$$= \frac{1}{1-x-y} \sum_{k \geq 0} (xy)^k \frac{\left(\frac{z}{1-x-y} \right)^k}{\left(1 - \frac{z}{1-x-y} \right)^{k+1}} = \frac{1}{1-x-y} \sum_{k \geq 0} \frac{\left(\frac{xyz}{1-x-y} \right)^k}{\left(\frac{1-x-y-z}{1-x-y} \right)^{k+1}}$$

$$\begin{aligned}
&= \frac{1}{1-x-y-z} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z} \right)^k = \frac{1}{1-x-y-z} \frac{1}{1 - \frac{xyz}{1-x-y-z}} \\
&= \frac{1}{1-x-y-z-xyz}. \quad \blacksquare
\end{aligned}$$

Proof 2:

$$\begin{aligned}
\sum_{p,q,r \geq 0} L_d(p,q,r) x^p y^q z^r &= \sum_{p,q,r \geq 0} \sum_{k \leq p,q,r} \binom{p+q+r-2k}{p-k, q-k, r-k, k} x^p y^q z^r \\
&= \sum_{k \geq 0} \sum_{p,q,r \geq k} \binom{p+q+r-2k}{p-k, q-k, r-k, k} x^{p-k} y^{q-k} z^{r-k} (xyz)^k \\
&= \sum_{n \geq 0} \sum_{k \geq 0} \sum_{\substack{p,q,r \geq k \\ p+q+r-2k=n}} \binom{n}{p-k, q-k, r-k, k} x^{p-k} y^{q-k} z^{r-k} (xyz)^k \\
&= \sum_{n \geq 0} \sum_{\substack{h,i,j,k \geq 0 \\ h+i+j+k=n \\ \text{iff} \\ p-k+q-k+r-k+k=n}} \binom{n}{h,i,j,k} x^h y^i z^j (xyz)^k \quad \text{Let } h=p-k, i=q-k, j=r-k. \\
&= \sum_{n \geq 0} (x+y+z+xyz)^n = \frac{1}{1-x-y-z-xyz}. \quad \blacksquare
\end{aligned}$$

Next, we will consider lattice paths that allow all four types of diagonal moves. The number of such lattice paths will be denoted $L_D(p, q, r)$. The following is a recursive formula for $L_D(p, q, r)$.

Theorem 4.7: For $p, q, r \in \mathbb{N}$, $L_D(p, 0, 0) = L_D(0, q, 0) = L_D(0, 0, r) = 1$. For $p, q, r \in \mathbb{P}$,
 $L_D(p, q, 0) = L_d(p, q)$, $L_D(p, 0, r) = L_d(p, r)$, and $L_D(0, q, r) = L_d(q, r)$. For
 $p, q, r, \in \mathbb{P}$,

$$L_D(p, q, r) = L_D(p, q, r-1) + L_D(p, q-1, r) + L_D(p-1, q, r) + L_D(p, q-1, r-1) \\ + L_D(p-1, q, r-1) + L_D(p-1, q-1, r) + L_D(p-1, q-1, r-1). \quad (4.2)$$

Proof: The only path from $(0, 0, 0)$ to $(p, 0, 0)$ is the one that consists of p X's. Hence,

$$L_D(p, 0, 0) = 1. \text{ Similarly, } L_D(0, q, 0) = L_D(0, 0, r) = 1.$$

Any lattice path from the point $(0, 0, 0)$ to the point $(p, q, 0)$ does not leave the xy -plane. Because xy -diagonal moves are allowed, these lattice paths correspond to the two-dimensional lattice paths from $(0, 0)$ to (p, q) in which diagonal moves are allowed. Thus, $L_D(p, q, 0) = L_d(p, q)$. Similarly,
 $L_D(p, 0, r) = L_d(p, r)$ and $L_D(0, q, r) = L_d(q, r)$.

The LHS of 4.2 counts the set of lattice paths from $(0, 0, 0)$ to (p, q, r) with all four diagonal moves allowed by definition. The RHS of 4.2 counts this set in seven disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p, q, r-1)$,
- ii. The category in which the last point visited before (p, q, r) is $(p, q-1, r)$,
- iii. The category in which the last point visited before (p, q, r) is $(p-1, q, r)$,
- iv. The category in which the last point visited before (p, q, r) is
 $(p, q-1, r-1)$,
- v. The category in which the last point visited before (p, q, r) is
 $(p-1, q, r-1)$,
- vi. The category in which the last point visited before (p, q, r) is
 $(p-1, q-1, r)$, and
- vii. The category in which the last point visited before (p, q, r) is
 $(p-1, q-1, r-1)$. ■

Now we want to find a closed form expression for $L_D(p, q, r)$.

Theorem 4.8: For $p, q, r \in \mathbb{N}$,

$$L_D(p, q, r) =$$

$$\sum_{d_{xy}, d_{xz}, d_{yz} \geq 0} \sum_{k=0}^{\min\{p,q,r\}} \binom{p+q+r-d_{xy}-d_{xz}-d_{yz}-2k}{p-k-d_{xy}-d_{xz}, q-k-d_{xy}-d_{yz}, r-k-d_{xz}-d_{yz}, d_{xy}, d_{xz}, d_{yz}, k}.$$

Proof: Suppose that a lattice path from $(0, 0, 0)$ to (p, q, r) has d_{xy} xy-diagonal moves, d_{xz} xz-diagonal moves, and d_{yz} yz-diagonal moves, and k cube diagonal moves. Such a path has $p - k - d_{xy} - d_{xz}$ moves parallel to the x-axis, $q - k - d_{xy} - d_{yz}$ moves parallel to the y-axis, and $r - k - d_{xz} - d_{yz}$ moves parallel to the z-axis. Just as we have represented moves parallel to the x-axis, y-axis, and z-axis, and cube diagonal moves as X, Y, Z, and D, we will represent xy-diagonal moves, xz-diagonal moves and yz-diagonal moves by A, B, and C respectively. Hence, the set of lattice paths with all four types of diagonal moves allowed is in one-to-one correspondence with the set of words that have $p - k - d_{xy} - d_{xz}$ X's, $q - k - d_{xy} - d_{yz}$ Y's, $r - k - d_{xz} - d_{yz}$ Z's, d_{xy} A's, d_{xz} B's, d_{yz} C's, and k D's. Hence, there are

$$\binom{p+q+r-d_{xy}-d_{xz}-d_{yz}-2k}{p-k-d_{xy}-d_{xz}, q-k-d_{xy}-d_{yz}, r-k-d_{xz}-d_{yz}, d_{xy}, d_{xz}, d_{yz}, k}$$

such lattice paths. ■

Theorem 4.9: The generating function for $L_D(p, q, r)$,

$$\sum_{p,q,r \geq 0} L_D(p, q, r) x^p y^q z^r = \frac{1}{1 - x - y - z - xy - xz - yz - xyz}.$$

Proof 1:
$$\sum_{p,q,r \geq 0} L_D(p, q, r) x^p y^q z^r$$

$$= \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \sum_{d_{xy}=0}^{\min\{p,q\}} \sum_{d_{xz}=0}^{\min\{q,r\}} \sum_{d_{yz}=0}^{\min\{p,r\}} \sum_{k=0}^{\min\{p,q,r\}}$$

$$\binom{p+q+r-d_{xy}-d_{xz}-d_{yz}-k}{p-k-d_{xy}-d_{xz}, q-k-d_{xy}-d_{yz}, r-k-d_{xz}-d_{yz}, d_{xy}, d_{xz}, d_{yz}, k} x^p y^q z^r$$

$$= \sum_{k \geq 0} \sum_{d_{yz} \geq 0} \sum_{d_{xz} \geq 0} \sum_{d_{xy} \geq 0} \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0}$$

$$\left(\begin{matrix} p+q+r-d_{xy}-d_{xz}-d_{yz}-k \\ p-k-d_{xy}-d_{xz}, q-k-d_{xy}-d_{yz}, r-k-d_{xz}-d_{yz}, d_{xy}, d_{xz}, d_{yz}, k \end{matrix} \right) x^p y^q z^r$$

$$= \sum_{k \geq 0} \sum_{d_{yz} \geq 0} \sum_{d_{xz} \geq 0} \sum_{d_{xy} \geq 0} \sum_{p \geq 0} \sum_{q \geq 0} \sum_{r \geq 0} \left(\begin{matrix} p+q+r-d_{xy}-d_{xz}-d_{yz}-2k \\ p-k-d_{xy}-d_{xz} \end{matrix} \right) \left(\begin{matrix} q+r-k-d_{yz} \\ q-k-d_{xy}-d_{yz} \end{matrix} \right)$$

$$\left(\begin{matrix} r+d_{xy} \\ r-k-d_{xz}-d_{yz} \end{matrix} \right) \left(\begin{matrix} k+d_{xy}+d_{xz}+d_{yz} \\ d_{xy} \end{matrix} \right) \left(\begin{matrix} k+d_{xz}+d_{yz} \\ d_{xz} \end{matrix} \right) \left(\begin{matrix} k+d_{yz} \\ d_{yz} \end{matrix} \right) x^p y^q z^r$$

$$= \sum_{k \geq 0} \sum_{d_{yz} \geq 0} \left(\begin{matrix} k+d_{yz} \\ d_{yz} \end{matrix} \right) \sum_{d_{xz} \geq 0} \left(\begin{matrix} k+d_{xz}+d_{yz} \\ d_{xz} \end{matrix} \right) \sum_{d_{xy} \geq 0} \left(\begin{matrix} k+d_{xy}+d_{xz}+d_{yz} \\ d_{xy} \end{matrix} \right)$$

$$\sum_{r \geq 0} \left(\begin{matrix} r+d_{xy} \\ r-k-d_{xz}-d_{yz} \end{matrix} \right) z^r \sum_{q \geq 0} \left(\begin{matrix} q+r-k-d_{yz} \\ q-k-d_{xy}-d_{yz} \end{matrix} \right) y^q$$

$$\sum_{p \geq 0} \left(\begin{matrix} p+q+r-d_{xy}-d_{xz}-d_{yz}-2k \\ p-k-d_{xy}-d_{xz} \end{matrix} \right) x^p$$

$$= \sum_{k \geq 0} x^k \sum_{d_{yz} \geq 0} \binom{k + d_{yz}}{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k + d_{xz} + d_{yz}}{d_{xz}} x^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k + d_{xy} + d_{xz} + d_{yz}}{d_{xy}} x^{d_{xy}}$$

$$\sum_{r \geq 0} \binom{r + d_{xy}}{r - k - d_{xz} - d_{yz}} z^r \sum_{q \geq 0} \binom{q + r - k - d_{yz}}{q - k - d_{xy} - d_{yz}} y^q$$

$$\sum_{p \geq k + d_{xy} + d_{xz}} \binom{p + q + r - d_{xy} - d_{xz} - d_{yz} - 2k}{p - k - d_{xy} - d_{xz}} x^{p - k - d_{xy} - d_{xz}}$$

$$= \sum_{k \geq 0} x^k \sum_{d_{yz} \geq 0} \binom{k + d_{yz}}{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k + d_{xz} + d_{yz}}{d_{xz}} x^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k + d_{xy} + d_{xz} + d_{yz}}{d_{xy}} x^{d_{xy}}$$

$$\sum_{r \geq 0} \binom{r + d_{xy}}{r - k - d_{xz} - d_{yz}} z^r \sum_{q \geq 0} \binom{q + r - k - d_{yz}}{q - k - d_{xy} - d_{yz}} y^q$$

$$\sum_{m \geq 0} \binom{m + q + r - d_{yz} - k}{m} x^m \quad \text{Let } m = p - k - d_{xy} - d_{xz}.$$

$$= \sum_{k \geq 0} x^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} x^{d_{xy}}$$

$$\sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} z^r \sum_{q \geq 0} \binom{q+r-k-d_{yz}}{q-k-d_{xy}-d_{yz}} y^q \left(\frac{1}{1-x} \right)^{q+r-d_{yz}-k+1}$$

$$= \frac{1}{1-x} \sum_{k \geq 0} x^k (1-x)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} (1-x)^{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}}$$

$$\sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} x^{d_{xy}} \sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} \left(\frac{z}{1-x} \right)^r$$

$$\sum_{q \geq 0} \binom{q+r-k-d_{yz}}{q-k-d_{xy}-d_{yz}} \left(\frac{y}{1-x} \right)^q$$

$$\begin{aligned}
&= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} y^{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}} \\
&\sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x} \right)^{d_{xy}} \sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} \left(\frac{z}{1-x} \right)^r \\
&\sum_{q \geq k+d_{xy}+d_{yz}} \binom{q+r-k-d_{yz}}{q-k-d_{xy}-d_{yz}} \left(\frac{y}{1-x} \right)^{q-k-d_{xy}-d_{yz}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} y^{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}} \\
&\sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x} \right)^{d_{xy}} \sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} \left(\frac{z}{1-x} \right)^r \\
&\sum_{n \geq 0} \binom{n+r+d_{xy}}{n} \left(\frac{y}{1-x} \right)^n \quad \text{Let } n = q - k - d_{xy} - d_{yz}.
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x} \sum_{k \geq 0} (xy)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} y^{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}} \\
&\sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x} \right)^{d_{xy}} \sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xy}-d_{yz}} \left(\frac{z}{1-x} \right)^r \left(\frac{1-x}{1-x-y} \right)^{r+d_{xy}+1} \\
&= \frac{1}{1-x-y} \sum_{k \geq 0} (xy)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} y^{d_{yz}} \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} x^{d_{xz}} \\
&\sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x-y} \right)^{d_{xy}} \sum_{r \geq 0} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} \left(\frac{z}{1-x-y} \right)^r \\
&= \frac{1}{1-x-y} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y} \right)^{d_{yz}} \\
&\sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y} \right)^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x-y} \right)^{d_{xy}} \\
&\sum_{r \geq k+d_{xz}+d_{yz}} \binom{r+d_{xy}}{r-k-d_{xz}-d_{yz}} \left(\frac{z}{1-x-y} \right)^{r-k-d_{xz}-d_{yz}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x-y} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y} \right)^{d_{yz}} \\
&\quad \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y} \right)^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x-y} \right)^{d_{xy}} \\
&\quad \sum_{s \geq 0} \binom{s+k+d_{xz}+d_{yz}+d_{xy}}{s} \left(\frac{z}{1-x-y} \right)^s \quad \text{Let } s = r - k - d_{xz} - d_{yz}.
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x-y} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y} \right)^{d_{yz}} \\
&\quad \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y} \right)^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x-y} \right)^{d_{xy}} \\
&\quad \left(\frac{1-x-y}{1-x-y-z} \right)^{k+d_{xy}+d_{xz}+d_{yz}+1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-x-y-z} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y-z} \right)^{d_{yz}} \\
&\quad \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y-z} \right)^{d_{xz}} \sum_{d_{xy} \geq 0} \binom{k+d_{xy}+d_{xz}+d_{yz}}{d_{xy}} \left(\frac{xy}{1-x-y-z} \right)^{d_{xy}} \\
&= \frac{1}{1-x-y-z} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y-z} \right)^{d_{yz}} \\
&\quad \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y-z} \right)^{d_{xz}} \left(\frac{1-x-y-z}{1-x-y-z-xy} \right)^{k+d_{xz}+d_{yz}+1} \\
&= \frac{1}{1-x-y-z-xy} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z-xy} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y-z-xy} \right)^{d_{yz}} \\
&\quad \sum_{d_{xz} \geq 0} \binom{k+d_{xz}+d_{yz}}{d_{xz}} \left(\frac{xz}{1-x-y-z-xy} \right)^{d_{xz}}
\end{aligned}$$

$$= \frac{1}{1-x-y-z-xy} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z-xy} \right)^k \sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y-z-xy} \right)^{d_{yz}}$$

$$\left(\frac{1-x-y-z-xy}{1-x-y-z-xy-xz} \right)^{k+d_{yz}+1}$$

$$= \frac{1}{1-x-y-z-xy-xz} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z-xy-xz} \right)^k$$

$$\sum_{d_{yz} \geq 0} \binom{k+d_{yz}}{d_{yz}} \left(\frac{yz}{1-x-y-z-xy-xz} \right)^{d_{yz}}$$

$$= \frac{1}{1-x-y-z-xy-xz} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z-xy-xz} \right)^k$$

$$\left(\frac{1-x-y-z-xy-xz}{1-x-y-z-xy-xz-yz} \right)^{k+1}$$

$$= \frac{1}{1-x-y-z-xy-xz-yz} \sum_{k \geq 0} \left(\frac{xyz}{1-x-y-z-xy-xz-yz} \right)^k$$

$$\begin{aligned}
&= \frac{1}{1-x-y-z-xy-xz-yz} \cdot \frac{1-x-y-z-xy-xz-yz}{1-x-y-z-xy-xz-yz-xyz} \\
&= \frac{1}{1-x-y-z-xy-xz-yz-xyz}.
\end{aligned}$$

Proof 2: $\sum_{p,q,r \geq 0} L_d(p,q,r) x^p y^q z^r$

$$= \sum_{p,q,r \geq 0} \sum_{d_{xy} \leq \min\{p,q\}} \sum_{d_{xz} \leq \min\{p,r\}} \sum_{d_{yz} \leq \min\{q,r\}} \sum_{k \leq p,q,r} \left(\begin{matrix} p+q+r-d_{xy}-d_{xz}-d_{yz}-2k \\ p-d_{xy}-d_{xz}-k, q-d_{xy}-d_{yz}-k, r-d_{xz}-d_{yz}-k, d_{xy}, d_{xz}, d_{yz}, k \end{matrix} \right) x^p y^q z^r$$

$$= \sum_{d_{xy} \geq 0} \sum_{d_{xz} \geq 0} \sum_{d_{yz} \geq 0} \sum_{k \geq 0} \sum_{p,q,r \geq d_{xy}, d_{xz}, d_{yz}, k} \left(\begin{matrix} p+q+r-d_{xy}-d_{xz}-d_{yz}-2k \\ p-d_{xy}-d_{xz}-k, q-d_{xy}-d_{yz}-k, r-d_{xz}-d_{yz}-k, d_{xy}, d_{xz}, d_{yz}, k \end{matrix} \right) x^p y^q z^r$$

$$\begin{aligned}
&= \sum_{n \geq 0} \sum_{d_{xy} \geq 0} \sum_{d_{xz} \geq 0} \sum_{d_{yz} \geq 0} \sum_{k \geq 0} \sum_{\substack{p,q,r \geq d_{xy}, d_{xz}, d_{yz}, k \\ p+q+r-d_{xy}-d_{xz}-d_{yz}-2k=n}} \left(\begin{matrix} n \\ p-d_{xy}-d_{xz}-k, q-d_{xy}-d_{yz}-k, r-d_{xz}-d_{yz}-k, d_{xy}, d_{xz}, d_{yz}, k \end{matrix} \right) \\
&\quad x^{p-d_{xy}-d_{xz}-k} y^{q-d_{xy}-d_{yz}-k} z^{r-d_{xz}-d_{yz}-k} (xy)^{d_{xy}} (xz)^{d_{xz}} (yz)^{d_{yz}} (xyz)^{d_{xyz}}
\end{aligned}$$

For $h = p - d_{xy} - d_{xz} - k$, $i = q - d_{xy} - d_{yz} - k$, and $j = r - d_{xz} - d_{yz} - k$.

$$\begin{aligned}
&= \sum_{n \geq 0} \sum_{\substack{h,i,j,k,d_{xy},d_{xz},d_{yz} \geq 0 \\ h+i+j+k+d_{xy}+d_{xz}+d_{yz}=n \\ \text{iff} \\ p+q+r-d_{xy}-d_{xz}-d_{yz}-2k=n}} \binom{n}{h,i,j,d_{xy},d_{xz},d_{yz},k} x^h y^i z^j (xy)^{d_{xy}} (xz)^{d_{xz}} (yz)^{d_{yz}} (xyz)^k \\
&= \sum_{n \geq 0} (x + y + z + xy + xz + yz + xyz)^n = \frac{1}{1 - x - y - z - xy - xz - yz - xyz}. \blacksquare
\end{aligned}$$

4.2. Restricted Paths: $L_d^*(p, q, r)$ and $L_n^*(p, q, r)$

As with lattice paths in $\mathbb{R}^3$ without diagonal moves allowed, many types of restrictions may be applied to lattice paths with diagonal moves allowed. We will examine two types of restricted lattice paths. First, we will consider lattice paths from $(0, 0, 0)$ to (p, q, r) with only cube diagonal moves allowed such that for each point (i, j, k) $i \geq j$. The number of these lattice paths will be denoted $L_d^*(p, q, r)$. The following is a recursive formula for $L_d^*(p, q, r)$.

Theorem 4.10 : For $p, r \in \mathbb{N}$, $L_d^*(p, 0, 0) = 1$ and $L_d^*(0, 0, r) = 1$. For $q \in \mathbb{P}$, $L_d^*(0, q, 0) = 0$. For $p \geq q$, $L_d^*(p, q, 0) = L^*(p, q)$. For $p, q, r \in \mathbb{P}$, $L_d^*(p, 0, r) = L(p, r)$ and $L_d^*(0, q, r) = 0$. If $p < q$, $L_d^*(p, q, r) = 0$. If $p \geq q$,

$$\begin{aligned}
L_d^*(p, q, r) &= L_d^*(p-1, q, r) + L_d^*(p, q-1, r) + L_d^*(p, q, r-1) \\
&\quad + L_d^*(p-1, q-1, r-1).
\end{aligned} \tag{4.3}$$

Proof: If $p, r \in \mathbb{N}$, the only path from $(0, 0, 0)$ to $(p, 0, 0)$ is the path that consists of p X 's, so $L_d^*(p, 0, 0) = 1$. Similarly, the only path from $(0, 0, 0)$ to $(0, 0, r)$ is the one that consists of r Z 's, so $L_d^*(0, 0, r) = 1$. If $q \in \mathbb{P}$, there is not a path from $(0, 0, 0)$ to $(0, q, 0)$ because $0 < q$.

Suppose $p \geq q$. Then any path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane; thus, the number of lattice paths with cube diagonal moves allowed such that for each lattice point $(i, j, 0)$ $i \geq j$ is the same as the number of lattice paths from $(0, 0)$ to (p, q) that are not allowed to cross the line $y = x$, $L^*(p, q)$.

Suppose $p, q, r \geq 1$. A lattice path from $(0, 0, 0)$ to $(p, 0, r)$ remains on the

xz-plane. The number of these lattice paths is the same as the number of lattice paths from $(0, 0)$ to (p, r) , $L(p, r)$. There are no lattice paths from $(0, 0, 0)$ to $(0, q, r)$ because $0 < q$; hence, $L_d^*(0, q, r) = 0$.

If $p < q$, then for the lattice path to travel from $(0, 0, 0)$ to (p, q, r) at some point (i, j, k) , specifically (p, q, r) , i must be less than j . Hence, $L_d^*(p, q, r) = 0$.

If $p \geq q$, then the LHS of 4.3 counts the set of lattice paths with cube diagonal moves allowed such that for each point (i, j, k) $i \geq j$ by definition. The RHS of 4.3 counts this set in four disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p - 1, q, r)$.
If $p = q$, this category is empty, and $L_d^*(p - 1, q, r) = 0$,
- ii. The category in which the last point visited before (p, q, r) is $(p, q - 1, r)$,
- iii. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$,
and
- iv. The category in which the last point visited before (p, q, r) is
 $(p - 1, q - 1, r - 1)$. ■

As before, we want a closed form expression for $L_d^*(p, q, r)$.

Theorem 4.11: For all $p, q, r \in \mathbb{N}$ with $p \geq q$,

$$L_d^*(p, q, r) = \sum_{k=0}^{\min\{q, r\}} \frac{p - q + 1}{p + q + r - 2k + 1} \binom{p + q + r - 2k + 1}{p - k + 1, q - k, r - k, k}.$$

Proof: Any lattice path with k cube diagonal moves from $(0, 0, 0)$ to (p, q, r) such that $i \geq j$ for each lattice point (i, j, k) on the path has $p - k$ moves parallel to the x-axis, $q - k$ moves parallel to the y-axis, and $r - k$ moves parallel to the z-axis. Thus, the corresponding word representation has $p - k$ X's, $q - k$ Y's, $r - k$ Z's, and k D's.

There are $\binom{p + q + r - 2k}{r - k}$ ways to place the Z's and then $\binom{p + q - k}{k}$ ways

to place the D's. We now want to place the X's and Y's so that at any point in the word there are at least as many Xs as there are Y's. By Theorem 1.7, there

are $\frac{p-q+1}{p-k+1} \binom{p+q-2k}{q-k}$ ways to place the X's and Y's. Hence,

$$\begin{aligned}
L_d^*(p, q, r) &= \sum_{k=0}^{\min\{q, r\}} \frac{p-q+1}{p-k+1} \binom{p+q-2k}{q-k} \binom{p+q-k}{k} \binom{p+q+r-2k}{r-k} \\
&= \sum_{k=0}^{\min\{q, r\}} \frac{p-q+1}{p-k+1} \frac{(p+q-2k)!}{(p-k)!(q-k)!} \frac{(p+q-k)!}{(p+q-2k)!k!} \frac{(p+q+r-2k)!}{(p+q-k)!(r-k)!} \\
&= \sum_{k=0}^{\min\{q, r\}} \frac{p-q+1}{p-k+1} \frac{(p+q+r-2k)!}{(p-k)!(q-k)!(r-k)!k!} \\
&= \sum_{k=0}^{\min\{q, r\}} \frac{p-q+1}{p+q+r-2k+1} \frac{(p+q+r-2k+1)!}{(p-k+1)!(q-k)!(r-k)!k!} \\
&= \sum_{k=0}^{\min\{q, r\}} \frac{p-q+1}{p+q+r-2k+1} \binom{p+q+r-2k+1}{p-k+1, q-k, r-k, k}. \quad \blacksquare
\end{aligned}$$

Secondly, we will consider those lattice paths from $(0, 0, 0)$ to (p, q, r) with xy-diagonal, xz-diagonal, yz-diagonal, and cube diagonal moves allowed such that for each lattice point (i, j, k) $i \geq j$. The number of these lattice paths will be denoted $L_D^*(p, q, r)$. The following is a recursive formula for $L_D^*(p, q, r)$.

Theorem 4.12: For $p \in \mathbb{N}$, $L_D^*(p, 0, 0) = 1$. For $q, r \in \mathbb{P}$, $L_D^*(0, q, 0) = 0$ and $L_D^*(0, 0, r) = 1$. For $p, q \in \mathbb{N}$ and $p \geq q$, $L_D^*(p, q, 0) = L_d^*(p, q)$. For $p, q, r \in \mathbb{P}$, $L_D^*(p, 0, r) = L_d(p, r)$ and $L_D^*(0, q, r) = 0$. If $p < q$, $L_D^*(p, q, r) = 0$. If $p \geq q$, $L_D^*(p, q, r) = L_D^*(p-1, q, r) + L_D^*(p, q-1, r) + L_D^*(p, q, r-1) + L_D^*(p-1, q-1, r) + L_D^*(p-1, q, r-1) + L_D^*(p, q-1, r-1) + L_D^*(p-1, q-1, r-1)$. (4.4)

Proof: From $(0, 0, 0)$ to $(p, 0, 0)$, the only path is the one that consists of p X's, so

$L_D^*(p, 0, 0) = 1$. From $(0, 0, 0)$ to $(0, q, 0)$, there is not a path because $0 < q$. Thus, $L_D^*(0, q, 0) = 0$. From $(0, 0, 0)$ to $(0, 0, r)$, the only path is the one that consists of r Z's. Thus, $L_D^*(0, 0, r) = 1$.

Suppose $p \geq q > 0$. Then, any lattice path from $(0, 0, 0)$ to $(p, q, 0)$ remains on the xy -plane. Thus, the number of lattice paths from $(0, 0, 0)$ to $(p, q, 0)$ with all four diagonal moves allowed such that for each lattice point $(i, j, 0)$ $i \geq j$ is the same as the number of lattice paths from $(0, 0)$ to (p, q) with diagonal moves allowed that do not cross the line $y = x$, $L_d^*(p, q)$.

Let $p, q, r \in \mathbf{P}$. Any lattice path from $(0, 0, 0)$ to $(p, 0, r)$ remains on the xz -plane. Thus, the number of such lattice paths is the same as the number of lattice paths from $(0, 0)$ to (p, r) with diagonal moves allowed, $L_d(p, r)$. There is not a lattice path from $(0, 0, 0)$ to $(0, q, r)$ because $0 < q$ so $L_D^*(0, q, r) = 0$.

Suppose $p < q$. Then for (p, q, r) $p < q$ and $L_D^*(p, q, r) = 0$.

Suppose $p \geq q$. Then, the LHS of 4.4 counts the set of lattice paths from $(0, 0, 0)$ to (p, q, r) with xy -diagonal, xz -diagonal, yz -diagonal, and cube diagonal moves allowed such that for each point (i, j, k) $i \geq j$ by definition. The RHS of 4.4 counts this set in seven disjoint exhaustive categories:

- i. The category in which the last point visited before (p, q, r) is $(p - 1, q, r)$.
If $p = q$, this category is empty, and $L_D^*(p - 1, q, r) = 0$.
- ii. The category in which the last point visited before (p, q, r) is $(p, q - 1, r)$.
- iii. The category in which the last point visited before (p, q, r) is $(p, q, r - 1)$,
- iv. The category in which the last point visited before (p, q, r) is $(p - 1, q - 1, r)$.
- v. The category in which the last point visited before (p, q, r) is $(p - 1, q, r - 1)$. If $p = q$, this category is empty, and $L_D^*(p - 1, q, r - 1) = 0$,
- vi. The category in which the last point visited before (p, q, r) is $(p, q - 1, r - 1)$, and
- vii. The category in which the last point visited before (p, q, r) is $(p - 1, q - 1, r - 1)$. ■

Theorem 4.13: For $p, q, r \in \mathbb{N}$,

$$L_D^*(p, q, r) = \sum_{d_{xy}, d_{xz}, d_{yz}, d_{xyz}} \frac{p - q - d_{xz} + d_{yz} + 1}{p + q + r - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz} + 1} \times \dots$$

$$\left(p - d_{xy} - d_{xz} - d_{xyz} + 1, q - d_{xy} - d_{yz} - d_{xyz}, r - d_{xz} - d_{yz} - d_{xyz}, d_{xy}, d_{xz}, d_{yz}, d_{xyz} \right).$$

Proof: A lattice path from $(0, 0, 0)$ to (p, q, r) with d_{xy} xy -diagonal moves, d_{xz} xz -diagonal moves, d_{yz} yz -diagonal moves, and d_{xyz} xyz -diagonal moves has $p - d_{xy} - d_{xz} - d_{xyz}$ moves parallel to the x -axis, $q - d_{xy} - d_{yz} - d_{xyz}$ moves parallel to the y -axis, and $r - d_{xz} - d_{yz} - d_{xyz}$ moves parallel to the z -axis. Thus, the corresponding word representation has $p - d_{xy} - d_{xz} - d_{xyz}$ X 's, $q - d_{xy} - d_{yz} - d_{xyz}$ Y 's, $r - d_{xz} - d_{yz} - d_{xyz}$ Z 's, d_{xy} A 's, d_{xz} B 's, d_{yz} C 's, and d_{xyz} D 's. There are

$$\left(p + q + r - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz}, p + q - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz}, r - d_{xz} - d_{yz} - d_{xyz}, d_{xz}, d_{yz}, d_{xyz} \right) \text{ ways to}$$

place the Z 's, B 's, C 's, and D 's. By Theorem 2.9, there are

$$\frac{p - q - d_{xy} + d_{yz} + 1}{p + q - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz} + 1} \left(p + q - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz} + 1, p - d_{xy} - d_{xz} - d_{xyz} + 1, q - d_{xy} - d_{yz} - d_{xyz}, d_{xyz} \right)$$

ways to place the remaining X 's, Y 's, and A 's so that at any point in the word representation there are at least as many X 's as Y 's. If we multiply these, we

obtain the desired result of $\frac{p - q - d_{xz} + d_{yz} + 1}{p + q + r - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz} + 1}$.

$$\left(p + q + r - d_{xy} - d_{xz} - d_{yz} - 2d_{xyz} + 1, p - d_{xy} - d_{xz} - d_{xyz} + 1, q - d_{xy} - d_{yz} - d_{xyz}, r - d_{xz} - d_{yz} - d_{xyz}, d_{xy}, d_{xz}, d_{yz}, d_{xyz} \right)$$

lattice paths with d_{xy} xy-diagonal, d_{xz} xz-diagonal, d_{yz} yz-diagonal, and d_{xyz} xyz-diagonal moves such that for each point (i, j, k) $i \geq j$. ■

4.3 Notes

Theorem 4.1 can be derived from the general formula in [4]. Theorem 4.2 appears in [5]. Theorems 4.6, 4.7, 4.8, and 4.9 appears in [2]. Theorems 4.11 and 4.13 are found in [5]. Theorem 4.12 can be derived from the general formula in [4].

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