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A Class of Functions That Are Quasiconvex But Not Polyconvex

Catherine S. Remus
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To the Graduate Council:

I am submitting herewith a thesis written by Catherine S. Remus entitled "A Class of Functions That Are Quasiconvex But Not Polyconvex." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

Henry C. Simpson, Major Professor

We have read this thesis and recommend its acceptance:

Charles Collins, G. Samuel Jordan

Accepted for the Council:
Carolyn R. Hodges

Vice Provost and Dean of the Graduate School

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A Class of Functions
That Are Quasiconvex But Not Polyconvex

A Thesis
Presented for the
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Catherine S. Remus
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Abstract

In 1991 V. Sverak [11] gave an example of a function that was invariant and quasiconvex but not polyconvex. We have generalized this example to a wide class of functions that meet certain ellipticity and growth conditions. Quasiconvexity is one necessary and sufficient condition for the existence of solutions to the minimization problem in elliptic P.D.E. theory. Invariance is frequently a requirement of the stored energy function in Calculus of Variation approaches to elasticity problems.

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Chapter I

Introduction

In this thesis a class of functions which are continuous, subject to certain growth and ellipticity conditions, and quasiconvex are shown not to be polyconvex. The study of convex analysis concerns itself with four types of convexity: convex functions, polyconvex functions, quasiconvex functions, and rank-one convex functions. The easiest way to visualize the relationships between these types of convexity is the analogy of nested sets where convex functions is the smallest, most stringent, set and rank-one convex functions is the largest, least stringent, set. The “proper subset” relationship of these convexity conditions has been a source of study for some years. In applications, quasiconvexity is very difficult to determine whereas polyconvexity is easier to show and the two concepts are near each other. V. Sverak [11] gave a new example of a function that is quasiconvex but not polyconvex in 1991. Other examples have been given in the past, but his new example was suggestive of an entire class of such functions. We construct a wide class of functions that are quasiconvex but not polyconvex. Sverak’s example is a special case of our examples.

Quasiconvexity is a necessary and sufficient condition (along with others) to ensure the existence of solutions to the minimization problem in elliptic P.D.E. theory. Techniques of variational calculus have long been utilized in elasticity problems and the stored energy function, our f, is often required to be objective, i.e., frame indifferent or Galilean invariant, and isotropic, i.e., rotationally invariant (see Dacorogna [5]).

In Chapter II we give background information such as definitions, notation, and a lemma needed in the proof of Theorem 1. Chapter III states and proves a needed preliminary theorem, Theorem 1, guaranteeing specific growth and quasiconvexity conditions on our class of functions. In Chapter IV we have the primary emphasis of this thesis, Theorem 2. This theorem deals with the nonpolyconvexity of a class of quasiconvex functions subject to subquadratic growth and ellipticity conditions. Chapter V presents Theorem 3 that addresses the invariance of functions in our class of quasiconvex functions. Finally, in Chapter VI we give four examples of functions in this class of functions we have constructed. Three of the four examples meet the criteria of all three of our theorems, but Example 2 is convex and therefore polyconvex and does not meet the criteria for Theorem 2. This alleged failure highlights the difficulty in constructing nontrivial examples of functions in this class satisfying all three theorems.

Chapter II

Background

In this thesis, $\mathbb{R}^+$ is the set of positive real numbers and $M^{n\times n}$ is the space of all $n\times n$ real matrices, $n\geq 1$, and $|\cdot|$ is any norm on this space. The 4-tensor, $\boldsymbol{C}$, is a linear transformation $M^{n\times n}\to M^{n\times n}$ and is of the general form $\boldsymbol{C}[X]_{ij}=\sum_{k,l=1}^{n}C_{ijkl}X_{kl}$, $C_{ijkl}\in\mathbb{R}$, for all $X\in M^{n\times n}$. For $\Omega\subset\mathbb{R}^n$, $\phi\in C^\infty(\Omega)$ with values in $\mathbb{R}^n$, we denote

$$\nabla\phi(x)=\begin{pmatrix}\frac{\partial\phi_1}{\partial x_1}(x) & \cdots & \frac{\partial\phi_1}{\partial x_n}(x)\\ \vdots & \ddots & \vdots\\ \frac{\partial\phi_n}{\partial x_1}(x) & \cdots & \frac{\partial\phi_n}{\partial x_n}(x)\end{pmatrix}, \text{ so that } \boldsymbol{C}\left[\nabla\phi(x)\right]_{ij}=\sum_{k,l=1}^{n}C_{ijkl}\frac{\partial\phi_k}{\partial x_l}(x).$$

For this work we will be considering functions $f:M^{n\times n}\to\mathbb{R}$ having various convexity properties. These are seen in physical theories such as elasticity, phase transition and many other applications that are considered the minimization of energy functionals, $\int_\Omega f\left(\nabla\phi(x)\right)dx$. The function f represents a potential energy function and ϕ is the deformation variable. There exist many theorems discussing this type of function, f, and all require some kind of convexity and associated conditions, e.g., growth and coercivity estimates (see Dacorogna [5]).

Definitions and Notation

$$\boldsymbol{C} \text{ is } \textbf{strongly elliptic} \text{ if and only if } \sum_{i,j,k,l=1}^{n} C_{ijkl} a_i b_j a_k b_l > 0 \quad \forall\, a,b \in \mathbb{R}^n \setminus \{\mathbf{0}\}$$

(Note this is a strengthened Legendre-Hadamard condition; the Legendre-Hadamard condition has $\geq$ in place of >.) (see Gelfand-Fomin [7], Wan [12], Morrey [9]).

Let $K \subset \mathbb{R}^n$. K is a **convex set** if, for any two points $x, y \in K$ and $t \in [0,1]$ the point $z = tx + (1-t)y$ is also in K.

Let $K \subset \mathbb{R}^n$. The **convex hull of** K, $co(K) :=$ the smallest convex set that contains K $= \{tx + (1-t)y \mid x, y \in K,\ t \in [0,1]\}$.

A function $g : \mathbb{R}^n \to \mathbb{R}$ is **convex** if and only if, for all $x, y \in \mathbb{R}^n$ and for all $t \in [0,1]$, $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$.

For the following several definitions, let $f : M^{n\times n} \to \mathbb{R}$ and let Ω be the open unit ball in $\mathbb{R}^n$.

The function f is **rank-one convex** if and only if, for every $A \in M^{n\times n}$ and for all $a, b \in \mathbb{R}^n$ the mapping $t \mapsto f(A + ta \otimes b)$, $t \in \mathbb{R}$ is convex. Here, $a \otimes b$ is the element of $M^{n\times n}$ with components $(a \otimes b)_{ij} = a_i b_j$, i.e., $a \otimes b = a\, b^{\mathrm{T}}$; this is the general rank one $n \times n$ matrix (provided either a or b is nonzero).

The function f is **quasiconvex** if and only if, for every $A \in M^{n\times n}$ and for every $\phi \in C_0^\infty(\Omega)$, $\int_\Omega f(A)\,dx \leq \int_\Omega f(A + \nabla\phi(x))\,dx$ is true.

The function f is **polyconvex** if and only if, for every $A \in M^{n \times n}$, there exists a convex function g mapping $\mathbb{R}^{\tau_n} \to \mathbb{R}$, such that $f(A) = g(T(A))$ where $T : M^{n \times n} \to \mathbb{R}^{\tau_n}$ and T is defined by $T(A) := (A,\ adj_2 A,\ \ldots,\ adj_n A)$. Here, $adj_s A$ is the adjugate matrix of all $s \times s$ minors of the matrix $A \in M^{n \times n}$, $2 \le s \le n$, and $\tau_n = \sum_{s=1}^{n} \sigma_s$ where $\sigma_s = \binom{n}{s}\binom{n}{s} = \left(\frac{n!}{s!(n-s)!} \right)^2$.

The function f is **convex** as defined above but with $M^{n \times n}$ replacing $\mathbb{R}^n$. The above definitions of convexity can be found in Dacorogna [5] and Ball [4].

It can be proven (see Dacorogna [5]) that f convex $\Rightarrow$ f polyconvex $\Rightarrow$ f quasiconvex $\Rightarrow$ f rank-one convex. It can also be shown that in the definition of quasiconvex, Ω may be replaced with any bounded open set in $\mathbb{R}^n$ and the resulting definition of quasiconvexity is equivalent to the one given above.

Next, we relate rank-one convexity to strong ellipticity as follows. Assuming $f \in C^2$, define a 4-tensor $\boldsymbol{C} : M^{n \times n} \to M^{n \times n}$ by $C_{ijkl} := \frac{\partial^2 f(A)}{\partial A_{ij} \partial A_{kl}}$. Again, it can be proven that if $\boldsymbol{C}$ is strongly elliptic for all $A \in M^{n \times n}$, then f is rank-one convex. Conversely, if f is rank-one convex, then $\boldsymbol{C}$ satisfies the Legendre-Hadamard condition for all $A \in M^{n \times n}$.

Let $f : M^{n\times n} \to \mathbb{R}$, $f \geq 0$, and let Ω be the open unit ball in $\mathbb{R}^n$.

The **convex envelope** or **convexification** of f is $Cf := \sup\{g \leq f \mid g \text{ is convex}\}$, i.e., $Cf(x) = \sup\{g(\ ) \mid g \leq f \text{ on } \Omega,\ g \text{ convex on } \Omega\}$, $x \in \Omega$; $Cf : M^{n\times n} \to \mathbb{R}$ is convex. The function Cf is the largest convex function below f, and f is convex if and only if $f = Cf$.

The **quasiconvex envelope** or **quasiconvexification** of f is $Qf := \sup\{g \leq f \mid g \text{ is quasiconvex}\}$; $Qf : M^{n\times n} \to \mathbb{R}$ is the largest quasiconvex function below f, and f is quasiconvex if and only if $f = Qf$. It can be shown that

$$Qf(A) := \inf\left\{\frac{1}{|\Omega|}\int_\Omega f\left(A + \nabla\phi(x)\right)dx \mid A \in M^{n\times n},\ \phi \in C_0^\infty(\Omega)\right\} \tag{2.1}$$

and $Cf \leq Qf \leq f$ (see Dacorogna [5]); also, if $f \geq c$, a constant, on $M^{n\times n}$ then $Cf \geq c$ on $M^{n\times n}$. Here, $|\Omega|$ is the volume of Ω.

We will need the following result from linear elliptic theory. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $1 < p < \infty$.

Let $\boldsymbol{C} : M^{n\times n} \to M^{n\times n}$ be a 4-tensor, $\boldsymbol{C}[X]_{ij} = \sum_{k,l=1}^{n} C_{ijkl} X_{kl}$, $C_{ijkl} \in \mathbb{R}$, for all $X \in M^{n\times n}$. We define, for $\phi \in C^\infty(\Omega)$, $div\boldsymbol{C}\left[\nabla\phi(x)\right]_i := \sum_{j,k,l=1}^{n} C_{ijkl}\frac{\partial^2\phi_k}{\partial x_l \partial x_j}(x)$, $x \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$.

As usual, we define, $L^p(\Omega) :=$ the class of all measurable functions $\psi : \Omega \to \mathbb{R}^n$ such that $|\psi|^p$ is integrable with the corresponding norm, $\|\psi\|_p := \left\{ \int_\Omega |\psi|^p \, dx \right\}^{1/p} < \infty$.

We also define $W^{1,p}(\Omega) := \left\{ \psi \in L^p(\Omega) \mid \nabla \psi \in L^p(\Omega) \right\}$ with its norm, $\|\psi\|_{W^{1,p}(\Omega)} := \left\{ \|\psi\|_{L^p(\Omega)}^p + \|\nabla \psi\|_{L^p(\Omega)}^p \right\}^{1/p}$. The space $W^{1,p}(\Omega)$ is a Banach space.

Define $W_0^{1,p}(\Omega) := \left\{ \psi \in W^{1,p}(\Omega) \mid \psi = 0 \text{ on } \partial\Omega \right\}$; $W_0^{1,p}(\Omega)$ is a closed subspace of $W^{1,p}(\Omega)$ and is therefore a Banach space with the same norm.

We define $W^{-1,q}(\Omega)$, $\left(\frac{1}{p} + \frac{1}{q} = 1 \right)$, to be the dual space to $W_0^{1,p}(\Omega)$, i.e., $W^{-1,q}(\Omega) := W_0^{1,p}(\Omega)^*$ (= space of bounded linear functionals on $W_0^{1,p}(\Omega)$), with the corresponding norm $\|\psi\|_{W^{-1,q}(\Omega)} := \sup_{\substack{u \in W_0^{1,p}(\Omega) \\ \|u\|_{W^{1,p}(\Omega)} \leq 1}} |\langle u, \psi \rangle|$ where $\langle u, \psi \rangle = \int_\Omega u(x) \psi(x) \, dx$.

If $\psi \in L^p(\Omega)$, then $\nabla \psi \in W^{-1,q}(\Omega)$ and also $\|\nabla \psi\|_{W^{-1,q}(\Omega)} \leq \|\psi\|_{L^p(\Omega)}$ (see Adams [1]). In particular,

$$\|div\boldsymbol{C}[\nabla \phi]\|_{W^{-1,q}(\Omega)} \leq \|\boldsymbol{C}[\nabla \phi]\|_{L^p(\Omega)}. \tag{2.2}$$

If we suppose that $\boldsymbol{C}$ is strongly elliptic, then it can be easily shown that the partial differential operator $div\boldsymbol{C}[\nabla \phi]$ is elliptic. To do this, we have that

$divC[\nabla\phi(x)]_i := \sum_{j,k,l=1}^{n} C_{ijkl}\frac{\partial^2\phi_k}{\partial x_l \partial x_j}(x)$, $x \in \mathbb{R}^n$. If we replace $\frac{\partial}{\partial x_j}$ with ξ_j where

$\xi = (\xi_1,\ \xi_2,\ \ldots,\ \xi_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, we arrive at the matrix $\mathrm{M} \in M^{n\times n}$, where

$\mathrm{M}_{ik} = \sum_{j,l=1}^{n} C_{ijkl}\xi_j\xi_l$. But M is positive definite, i.e., $\sum_{i,k=1}^{n}\mathrm{M}_{ik}\eta_i\eta_k > 0$ for all

$\eta = (\eta_1,\ \eta_2,\ \ldots,\ \eta_n) \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, i.e., $\eta^{\mathsf{T}}\mathrm{M}\eta > 0$ for all $\eta \neq 0$, since $\boldsymbol{C}$ is strongly elliptic. Then M is nonsingular. Thus, $div\boldsymbol{C}[\nabla\phi(x)]$ is elliptic (see Agmon, Douglis, Nirenberg [3] or Morrey [9]). Then we have the following result.

Morrey's Lemma

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 < p < \infty$, and let $\boldsymbol{C}$ be a strongly elliptic 4-tensor. Then $\exists\ c > 0$ such that $c\|\phi\|_{W^{1,p}(\Omega)} \leq \|div\boldsymbol{C}[\nabla\phi]\|_{W^{-1,q}(\Omega)}\ \forall\ \phi \in W_0^{1,p}(\Omega)$,

$\frac{1}{p} + \frac{1}{q} = 1$. In particular,

$$c\|\phi\|_{W^{1,p}(\Omega)} \leq \|\boldsymbol{C}[\nabla\phi]\|_{L^p(\Omega)} \tag{2.3}$$

$\forall\ \phi \in W_0^{1,p}(\Omega)$.

Proof

The Lemma is essentially in Agmon, Douglis, Nirenberg [2] or Morrey [9] (see also Simpson [10]).

In Agmon, Douglis, Nirenberg [2] see Theorem 15.1 page 707. We will write the operator in "integral form" $\mathrm{L}\phi = div\boldsymbol{C}[\nabla\phi] = \sum_{j=1}^{n}\frac{\partial}{\partial x_j} f_j$ where $f_j = \boldsymbol{C}[\nabla\phi]_{ij}$. In that paper, $l=1$, $m=n$, $\mathfrak{D}=$ any bounded open set containing Ω such that $\overline{\Omega}\subset\mathfrak{D}$, and $\mathfrak{U}=\Omega$. Then the theorem in Agmon, Douglis, Nirenberg [2] states that

$$\sum_{\mu=1}^{n}\int_{\Omega}\left|\frac{\partial\phi}{\partial x_{\mu}}\right|^{p}dx\leq \text{ constant}\int_{\mathfrak{D}}\left(\sum_{\beta=1}^{n}\left|f_{\beta}\right|^{p}+\left|\phi\right|^{p}\right)dx$$

for all $\phi\in W^{1,p}\left(\mathfrak{D}\right)$. This yields

$$c_1\left\|\phi\right\|_{W^{1,p}(\Omega)}\leq\left\|\boldsymbol{C}\left[\nabla\phi\right]\right\|_{L^p(\mathfrak{D})}+\left\|\phi\right\|_{L^p(\mathfrak{D})}\tag{2.4}$$

for all $\phi\in W^{1,p}\left(\mathfrak{D}\right)$. To prove (2.3) from (2.4), assume for a contradiction, that (2.3) fails. Then, there exists a sequence, $\left\{\phi_j\right\}_{j=1}^{\infty}\subset W_0^{1,p}\left(\Omega\right)$, such that $\left\|\phi_j\right\|_{W^{1,p}(\Omega)}=1$ for all j and $\left\|\boldsymbol{C}\left[\nabla\phi_j\right]\right\|_{L^p(\Omega)}\rightarrow 0$ as $j\rightarrow\infty$. By Rellich compactness $W_0^{1,p}\left(\Omega\right)\rightarrow L^p\left(\Omega\right)$ is a compact imbedding (see Adams [1]). So there exists a subsequence, $\left\{\phi_{j_i}\right\}$, that converges strongly in $L^p\left(\Omega\right)$. We will relabel ϕ_{j_i} as ϕ_j. Let $\phi_j\rightarrow\phi$ strongly in $L^p\left(\Omega\right)$. From (2.4) we have

$$\begin{aligned}c_1\left\|\phi_j-\phi_k\right\|_{W^{1,p}(\Omega)}&\leq\left\|\boldsymbol{C}\left[\nabla\phi_j\right]-\boldsymbol{C}\left[\nabla\phi_k\right]\right\|_{L^p(\Omega)}+\left\|\phi_j-\phi_k\right\|_{L^p(\Omega)}\\&\leq\left\|\boldsymbol{C}\left[\nabla\phi_j\right]\right\|_{L^p(\Omega)}+\left\|\boldsymbol{C}\left[\nabla\phi_k\right]\right\|_{L^p(\Omega)}+\left\|\phi_j-\phi_k\right\|_{L^p(\Omega)}\quad\text{(by Triangle Inequality).}\end{aligned}$$

Then, $\left\{\phi_j\right\}$ is Cauchy in $W^{1,p}\left(\Omega\right)$. So $\phi_j\rightarrow\phi$ in $W_0^{1,p}\left(\Omega\right)$ where $\phi\in W_0^{1,p}\left(\Omega\right)$. Thus, $\boldsymbol{C}\left[\nabla\phi\right]=0$ a.e. in Ω. Therefore, $div\boldsymbol{C}\left[\nabla\phi\right]=0$ a.e. in Ω (weak solution). By elliptic regularity theory (see Morrey [9]), $\phi\in C^{\infty}\left(\overline{\Omega}\right)$.

We now prove that $\phi \equiv 0$ in Ω. First extend ϕ to be identically 0 on $\mathbb{R}^n \setminus \Omega$, so $\phi \in W^{1,2}\left(\mathbb{R}^n\right)$. The Fourier Transform of ϕ is

$\hat{\phi}(\xi) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} e^{-i(x \bullet \xi)} \phi(x)\, dx, \;\; x, \xi \in \mathbb{R}^n$. Then, the Fourier Transform of $\nabla\phi$ is

$\widehat{\nabla\phi}(\xi) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} e^{-i(x \bullet \xi)} \nabla\phi(x)\, dx = i\hat{\phi}(\xi) \otimes \xi \qquad$ (by integration by parts) . Then,

$\boldsymbol{C}[\nabla\phi]_{ij} = \displaystyle\sum_{k,l=1}^{n} C_{ijkl} \frac{\partial \phi_k}{\partial x_l} = 0$ yields $\boldsymbol{C}\left[\hat{\phi}(\xi) \otimes \xi\right] = 0$ a.e. for $\xi \in \mathbb{R}^n$. Further,

$\left(\hat{\phi}(\xi) \otimes \xi\right) \bullet \boldsymbol{C}\left[\hat{\phi}(\xi) \otimes \xi\right] = 0$, or $\displaystyle\sum_{ijkl=1}^{n} C_{ijkl} \hat{\phi}_i(\xi) \xi_j \hat{\phi}_k(\xi) \xi_l = 0$ a.e. for $\xi \in \mathbb{R}^n$. By the strong ellipticity hypothesis for $\boldsymbol{C}$, $\hat{\phi}(\xi) = 0$ a.e. for $\xi \in \mathbb{R}^n$. Thus, by Fourier Transform Theory, $\phi \equiv 0$ a.e. in $\mathbb{R}^n$. But $\left\|\phi_j\right\|_{W^{1,p}(\Omega)} = 1 \Rightarrow \left\|\phi\right\|_{W^{1,p}(\Omega)} = 1$. Since, $\phi \equiv 0$, we have the desired contradiction and, therefore, (2.3) holds.

Alternatively, in Morrey [9], see Theorem 6.4.4 page 246. There, $\upsilon = N = n$, $t_k = 2$, $s_j = 0$, $m_j = 1$, $\rho_{jk} = 1$, and $h = h_0 = -1$. Let B denote any open ball containing $\bar{\Omega}$ and let $u \in W_0^{1,p}(B)$. Also, $a_{jk}^{\alpha\beta} = C_{j\alpha k\beta}$ for $|\alpha| = |\beta| = 1$, $a_{jk}^{\alpha\beta} = 0$ for $|\alpha| = 0$ or $|\beta| = 0$; $f_j^{\alpha} = \boldsymbol{C}[\nabla u]_{j\alpha}$ for $|\alpha| = 1$, $f_j^{\alpha} = 0$ for $|\alpha| = 0$. This is a solution to the weak formulation $\displaystyle\int_B \sum_{j=1}^{N} \sum_{|\alpha| \le 1} D^{\alpha} \upsilon_j \left[\sum_{k=1}^{n} \sum_{|\beta| \le 1} a_{jk}^{\alpha\beta} D^{\beta} u^k - f_j^{\alpha} \right] dx = 0 \;\; \forall \; \upsilon \in W_0^{1,p}(B)$ (see equation (6.4.1) on page 242 Morrey [9]). Then, by the theorem cited, we get the estimate

$$\begin{aligned}\|u\|_{W^{1,p}(B)} &\leq c\left[\|f\|_{L^p(B)}+\|u\|_{L^1(B)}\right]\\ &\leq c'\left[\|f\|_{L^p(B)}+\|u\|_{L^p(B)}\right].\end{aligned}$$

Then this yields (2.4) again when identifying $\mathfrak{D}$ with B. Therefore, as before, (2.3) follows. ■

Chapter III

Theorem 1

Purpose

We wish to construct a class of quasiconvex functions on $M^{2\times 2}$ which are not polyconvex. In order to achieve this we need the preliminary Theorem 1, ensuring certain specific conditions on this class of functions. First we look at a class of functions $f : M^{n\times n} \to \mathbb{R}$ which are bounded above and below in a particular way. Then we look at f 's quasiconvexification, Qf .

Theorem 1

Let $1 < p < \infty$ and $f : M^{n\times n} \to \mathbb{R}$, be continuous, such that

$\left|\boldsymbol{C}\left[X\right]\right|^p \le f\left(X\right) \le c_1 + c_2\left|X\right|^p \ \ \forall\, X \in M^{n\times n}$ where $c_1, c_2 \in \mathbb{R}^+$ and $\boldsymbol{C}$ is a strongly elliptic 4-tensor. Let $L = \left\{X \in M^{n\times n} \mid \boldsymbol{C}\left[X\right] = 0\right\}$ and $K = \left\{X \in M^{n\times n} \mid f\left(X\right) = 0\right\}$; let $\Omega \subset \mathbb{R}^n$ be the open unit ball. Then the quasiconvexification of f , Qf , satisfies $0 < Qf\left(X\right)$ for any $X \in M^{n\times n} \setminus K$.

Proof

Note that $K \subset L$. Using the relationship, $Cf \le Qf$ (see Dacorogna [5]) we first show Qf is positive on $M^{n\times n} \setminus L$. To do this, it suffices to demonstrate that $\left|\boldsymbol{C}\left[X\right]\right|^p$ is convex to get $0 < \left|\boldsymbol{C}\left[X\right]\right|^p \le Cf\left(X\right) \le Qf\left(X\right) \ \ \forall\, X \in M^{n\times n} \setminus L$.

To show $\left|\boldsymbol{C}[X]\right|^p$ is convex, let $X, Y \in M^{n\times n}$, $t \in [0,1]$. Then,

$$\begin{aligned}\left|\boldsymbol{C}\left[tX+(1-t)Y\right]\right|^p &= \left|t\boldsymbol{C}[X]+(1-t)\boldsymbol{C}[Y]\right|^p \\ &= \left|t_1\boldsymbol{C}[X]+t_2\boldsymbol{C}[Y]\right|^p,\ t_1 = t \text{ and } t_2 = (1-t) \\ &\le \left[t_1\left|\boldsymbol{C}[X]\right|+t_2\left|\boldsymbol{C}[Y]\right|\right]^p,\ \text{by the Triangle Inequality} \\ &= \left[t_1^{1/q}\left(t_1^{1/p}\left|\boldsymbol{C}[X]\right|\right)+t_2^{1/q}\left(t_2^{1/p}\left|\boldsymbol{C}[Y]\right|\right)\right]^p,\ p,q>1 \text{ s.t. } \frac{1}{p}+\frac{1}{q}=1.\end{aligned} \tag{3.1}$$

Holder's Inequality states

$a_1b_1 + a_2b_2 \le \left(a_1^q + a_2^q\right)^{1/q}\left(b_1^p + b_2^p\right)^{1/p}$, where $p,q>1$ s.t. $\frac{1}{p}+\frac{1}{q}=1$. If we let

$a_1 = t_1^{1/q}$, $b_1 = t_1^{1/p}\left|\boldsymbol{C}[X]\right|$, $a_2 = t_2^{1/q}$, and $b_2 = t_2^{1/p}\left|\boldsymbol{C}[Y]\right|$, then

$$t_1^{1/q}\left(t_1^{1/p}\left|\boldsymbol{C}[X]\right|\right)+t_2^{1/q}\left(t_2^{1/p}\left|\boldsymbol{C}[Y]\right|\right) \le (t_1+t_2)^{1/q}\left(t_1\left|\boldsymbol{C}[X]\right|^p+t_2\left|\boldsymbol{C}[Y]\right|^p\right)^{1/p}. \tag{3.2}$$

Using (3.2) in (3.1) we have

$$\begin{aligned}\left|\boldsymbol{C}\left[tX+(1-t)Y\right]\right|^p &\le \left[t_1^{1/q}\left(t_1^{1/p}\left|\boldsymbol{C}[X]\right|\right)+t_2^{1/q}\left(t_2^{1/p}\left|\boldsymbol{C}[Y]\right|\right)\right]^p \\ &\le \left[(t_1+t_2)^{1/q}\left(t_1\left|\boldsymbol{C}[X]\right|^p+t_2\left|\boldsymbol{C}[Y]\right|^p\right)^{1/p}\right]^p \\ &= \left[(t+(1-t))^{1/q}\left(t\left|\boldsymbol{C}[X]\right|^p+(1-t)\left|\boldsymbol{C}[Y]\right|^p\right)^{1/p}\right]^p \\ &= \left[1^{1/q}\left(t\left|\boldsymbol{C}[X]\right|^p+(1-t)\left|\boldsymbol{C}[Y]\right|^p\right)^{1/p}\right]^p \\ &= t\left|\boldsymbol{C}[X]\right|^p+(1-t)\left|\boldsymbol{C}[Y]\right|^p.\end{aligned}$$

Therefore $\left|\boldsymbol{C}\left[tX+(1-t)Y\right]\right|^p \le t\left|\boldsymbol{C}\left[X\right]\right|^p+(1-t)\left|\boldsymbol{C}\left[Y\right]\right|^p$ which gives us that $\left|\boldsymbol{C}\left[X\right]\right|^p$ is convex $\forall\ X\in M^{n\times n}$. Thus, $\forall\ X\in M^{n\times n}\setminus L,\quad 0<Qf\left(X\right)$ is true.

It now remains to be shown that $0<Qf\left(X\right)\ \forall\ X\in L\setminus K.$ Suppose, for a contradiction, that $Qf\left(A\right)=0$ for some $A\in L\setminus K.$ By (2.1) $\exists$ a sequence $\phi_j\in C_0^{\infty}\left(\Omega\right)$ such that $\lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega f\left(A+\nabla\phi_j\left(x\right)\right)dx=0.$ We know that

$\forall\ X\in M^{n\times n}\setminus K\qquad \left|\boldsymbol{C}\left[X\right]\right|^p\le Cf\left(X\right)\le Qf\left(X\right)\le f\left(X\right).$ Then,

$\lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega \left|\boldsymbol{C}\left[A+\nabla\phi_j\left(x\right)\right]\right|^p dx\le \lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega f\left(A+\nabla\phi_j\left(x\right)\right)dx=0$ which implies

$\lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega \left|\boldsymbol{C}\left[A+\nabla\phi_j\left(x\right)\right]\right|^p dx=0.$

Recall, since $A\in L$, $\boldsymbol{C}\left[A+\nabla\phi_j\left(x\right)\right]=\boldsymbol{C}\left[A\right]+\boldsymbol{C}\left[\nabla\phi_j\left(x\right)\right]=\boldsymbol{C}\left[\nabla\phi_j\left(x\right)\right].$

Therefore, $\lim_{j\to\infty}\int_\Omega \left|\boldsymbol{C}\left[\nabla\phi_j\left(x\right)\right]\right|^p dx=0.$ By Morrey's Lemma (see Chapter II), $\lim_{j\to\infty}\left\|\phi_j\right\|_{W^{1,p}(\Omega)}=0$. There is a subsequence of ϕ_j 's which we will relabel as ϕ_j itself such that $\phi_j\to 0$ *a.e.* in Ω (see Hewitt and Stromberg [8, Theorem 13.11]).

Now we use Fatou's Lemma, which states, let $\{g_n\}$ be a sequence of nonnegative integrable functions over Ω and let $g\left(x\right)=\liminf_{j\to\infty} g_j\left(x\right)$; then, $\int_\Omega g\ dx\le\liminf_{j\to\infty}\int_\Omega g_j\ dx$

(see Friedman [6]). Now, if $g_j = f\left(A+\nabla\phi_j\right)$, then $0 \le \left|\boldsymbol{C}\left[\nabla\phi_j\right]\right|^p \le f\left(A+\nabla\phi_j\right)$ implies $0 \le g_j$. Also, $f\left(A+\nabla\phi_j\right) \le c_1 + c_2\left|A+\nabla\phi_j\right|^p \le c_1 + c_3\left(\left|A\right|^p + \left|\nabla\phi_j\right|^p\right)$ for some constant, $c_3 > 0$ implies g_j is integrable over Ω and f continuous implies $\lim_{j\to\infty} f\left(A+\nabla\phi_j\right) = f\left(A + \lim_{j\to\infty}\nabla\phi_j\right) = f\left(A\right)$ *a.e.* in Ω, which implies $g = f\left(A\right)$.

By Fatou's Lemma, we have

$$\begin{aligned}\liminf_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega f\left(A+\nabla\phi_j\right)dx &\ge \frac{1}{|\Omega|}\int_\Omega \lim_{j\to\infty} f\left(A+\nabla\phi_j\right)dx \\ &= \frac{1}{|\Omega|}\int_\Omega f\left(A\right)dx \\ &= f\left(A\right).\end{aligned}$$

By our hypothesis, $0 = \lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega f\left(A+\nabla\phi_j\right)dx$, so we have $0 \ge f\left(A\right)$. However, $0 = \left|\boldsymbol{C}\left[A\right]\right|^p \le f\left(A\right) \Rightarrow 0 \le f\left(A\right)$. Thus, $0 \le f\left(A\right) \le 0$, so that $f\left(A\right) = 0$ and $A \in K$ which is the desired contradiction. Thus, $Qf\left(X\right) > 0 \;\; \forall\, X \in L \setminus K.$ We conclude $Qf > 0$ on $M^{n\times n} \setminus K$. ■

Chapter IV

Theorem 2

<u>Purpose</u>

Now that we have established the needed preliminary theorem for quasiconvex functions on $M^{n\times n}$ and for $1 < p < \infty$, we will address the issue of the polyconvexity of Qf on $M^{2\times 2}$ but with $1 < p < 2$.

<u>Theorem 2</u>

Let $1 < p < 2,\ \ f : M^{2\times 2} \to \mathbb{R}$ be continuous and satisfy $\left|\boldsymbol{C}[X]\right|^p \le f(X) \le c_1 + c_2 |X|^p \ \ \forall\, X \in M^{2\times 2}$ where $c_1, c_2 \in \mathbb{R}^+$ and $\boldsymbol{C}$ is a strongly elliptic 4-tensor. Assume $K = \left\{X \in M^{2\times 2} \mid f(X) = 0\right\}$ is nonconvex. Then the quasiconvexification of f, Qf, is quasiconvex but not polyconvex.

Proof

Qf is quasiconvex by definition. Suppose, for a contradiction, that Qf is polyconvex. Since $Qf \le f$, then Qf has sub-quadratic growth at ∞, i.e., $Qf(X) \le c_1 + c_2 |X|^p \ \ \forall\, X \in M^{2\times 2}$. It is known that every polyconvex function in $M^{2\times 2}$ with subquadratic growth at ∞ is convex (see Sverak [11] or Simpson [10]). Next, $0 \le Qf \le f$ implies $Qf = 0$ on K. Therefore, $Qf = 0$ on $co(K) =$ convex hull of K. But K is nonconvex, so there are points in $co(K)$ that are not in K. We now have a contradiction to Theorem 1, and therefore, the desired contradiction, i.e., Qf is not polyconvex. ■

Chapter V

Theorem 3

Purpose

We will now discuss the invariance of this class of quasiconvex, but nonpolyconvex functions. This class of functions is important to applications involving elasticity of materials, phase transitions, and other applications. Since polyconvexity implies quasiconvexity and polyconvexity is easier to verify and the gap between the two is slight, our construction of this class of functions that are quasiconvex but not polyconvex is mathematically important. Recall, $O(2)$ is the set of orthogonal 2×2 matrices, R, such that $\det R = \pm1$ and $SO(2)$ is the set of orthogonal 2×2 matrices, R^+, such that $\det R^+ = +1$ where R^+ denotes a proper rotation.

Theorem 3

Let $1 < p < 2$, $f : M^{2\times2} \to \mathbb{R}$ be continuous and satisfy $\left|\boldsymbol{C}\left[X\right]\right|^p \le f\left(X\right) \le c_1 + c_2\left|X\right|^p \;\; \forall\, X \in M^{2\times2}$ where $c_1, c_2 \in \mathbb{R}^+$ and $\boldsymbol{C}$ is a strongly elliptic 4-tensor. Assume f is invariant with respect to $O(2)$, i.e., $f\left(RXR^{\mathsf{T}}\right) = f\left(X\right)$ $\forall\, X \in M^{2\times2}$ and $\forall\, R \in O(2)$, and assume $K = \left\{X \in M^{2\times2} \mid f\left(X\right) = 0\right\}$ is nonconvex. Then the quasiconvexification of f, Qf, is invariant with respect to $O(2)$ and quasiconvex but not polyconvex. Similarly, suppose $f\left(R^+X\right) = f\left(X\right)$ $\forall\, X \in M^{2\times2}$ and

$\forall\ R^+ \in SO(2)$, and assume $K = \left\{X \in M^{2\times 2} \mid f(X) = 0\right\}$ is nonconvex. Then the quasiconvexification of f, Qf, is invariant with respect to $SO(2)$ and quasiconvex but not polyconvex.

Proof

We suppose $\Omega \subset \mathbb{R}^2$ is the unit ball and $f\left(RXR^\intercal\right) = f(X)\ \ \forall\ X \in M^{2\times 2}$ and $\forall\ R \in O(2)$. Recall, $O(2)$ is the set of orthogonal 2×2 matrices, R, such that $\det R = \pm 1$. We show that Qf is invariant, i.e., $Qf\left(RXR^\intercal\right) = Qf(X)$. Fix $R \in O(2)$ and fix $X \in M^{2\times 2}$. We have $Qf\left(RXR^\intercal\right) = \inf\left\{\frac{1}{|\Omega|}\int_\Omega f\left(RXR^\intercal + \nabla\phi(x)\right)dx \mid \ \phi \in C_0^\infty(\Omega)\right\}$.

Note that, $f\left(RXR^\intercal + \nabla\phi(x)\right) = f\left(R\left(X + R^\intercal\nabla\phi(x)R\right)R^\intercal\right) = f\left(X + R^\intercal\nabla\phi(x)R\right)$, by the invariance of f. This implies

$$Qf\left(RXR^\intercal\right) = \inf\left\{\frac{1}{|\Omega|}\int_\Omega f\left(X + R^\intercal\nabla\phi(x)R\right)dx \mid \ \phi \in C_0^\infty(\Omega)\right\}. \tag{5.1}$$

We make a change of variable in the last integral and let $x = R\tilde{x}$. This transformation does not affect Ω. Then,

$$\begin{aligned} dx &= \left|\frac{\partial x}{\partial \tilde{x}}\right| d\tilde{x} \\ &= \left|\det R\right| d\tilde{x} \\ &= \left|\pm 1\right| d\tilde{x} = d\tilde{x} \end{aligned} \quad , \quad \frac{\partial x}{\partial \tilde{x}} = \begin{pmatrix} \frac{\partial x_1}{\partial \tilde{x}_1} & \frac{\partial x_1}{\partial \tilde{x}_2} \\ \frac{\partial x_2}{\partial \tilde{x}_1} & \frac{\partial x_2}{\partial \tilde{x}_2} \end{pmatrix} = R$$

and equation (5.1) becomes

$Qf\left(RXR^{\intercal}\right)=\inf\left\{\frac{1}{|\Omega|}\int_{\Omega} f\left(X+R^{\intercal}\nabla\phi\left(R\tilde{x}\right)R\right)d\tilde{x} \mid \phi\in C_0^{\infty}\left(\Omega\right)\right\}$. We now let

$\psi\left(\tilde{x}\right)=R^{\intercal}\phi\left(R\tilde{x}\right)$, $\tilde{x}\in\Omega$, $\psi\in C_0^{\infty}\left(\Omega\right)$. Next, we show $\tilde{\nabla}\psi\left(\tilde{x}\right)=R^{\intercal}\nabla\phi\left(R\tilde{x}\right)R$ where $\tilde{\nabla}$ is the matrix of partial derivatives with respect to $\tilde{x}$. In fact,

$$\begin{aligned}
\left[\tilde{\nabla}\psi\left(\tilde{x}\right)\right]_{ij} &= \frac{\partial\psi_i\left(\tilde{x}\right)}{\partial\tilde{x}_j} \\
&= \frac{\partial}{\partial\tilde{x}_j}\left(\sum_{k=1}^{2} R^{\intercal}{}_{ik}\phi_k\left(R\tilde{x}\right)\right) \\
&= \sum_{k=1}^{2} R^{\intercal}{}_{ik}\frac{\partial\phi_k}{\partial x_l}\frac{\partial x_l}{\partial\tilde{x}_j} \\
&= \sum_{k=1}^{2} R^{\intercal}{}_{ik}\frac{\partial\phi_k}{\partial x_l}R_{lj} \\
&= \left[R^{\intercal}\nabla\phi\left(R\tilde{x}\right)R\right]_{ij}.
\end{aligned}$$

We now have $Qf\left(RXR^{\intercal}\right)=\inf\left\{\frac{1}{|\Omega|}\int_{\Omega} f\left(X+\tilde{\nabla}\psi\left(\tilde{x}\right)\right)d\tilde{x} \mid \psi\in C_0^{\infty}\left(\Omega\right)\right\}=Qf\left(X\right)$.

Thus, Qf is invariant as desired. Then we have, by Theorem 2, Qf is quasiconvex, invariant but not polyconvex.

Now, suppose $f\left(R^{+}X\right)=f\left(X\right)$ $\forall X\in M^{2\times2}$ and $\forall R^{+}\in SO(2)$. We again recall, $SO(2)=$ the set of orthogonal 2×2 matrices, R^{+}, such that $\det R^{+}=+1$ where R^{+} denotes a proper rotation. We show that Qf is invariant, i.e., $Qf\left(R^{+}X\right)=Qf\left(X\right)$. Fix $R\in SO(2)$ and fix $X\in M^{2\times2}$. We then have

$Qf\left(R^{+}X\right)=\inf\left\{\frac{1}{|\Omega|}\int_{\Omega} f\left(R^{+}X+\nabla\phi(x)\right)dx \mid \phi\in C_0^{\infty}(\Omega)\right\}$. Note that,

$f\left(R^{+}X+\nabla\phi(x)\right)=f\left(R^{+}\left(X+\left(R^{+}\right)^{\mathsf{T}}\nabla\phi(x)\right)\right)=f\left(X+\left(R^{+}\right)^{\mathsf{T}}\nabla\phi(x)\right)$ as before. We now let $\psi(x)=\left(R^{+}\right)^{\mathsf{T}}\phi(x)$, then $\nabla\psi(x)=\nabla\left(\left(R^{+}\right)^{\mathsf{T}}\phi(x)\right)=\left(R^{+}\right)^{\mathsf{T}}\nabla\phi(x)$. This yields

$$\begin{aligned} Qf\left(R^{+}X\right)&=\inf\left\{\frac{1}{|\Omega|}\int_{\Omega} f\left(X+\nabla\psi(x)\right)dx \mid \psi\in C_0^{\infty}(\Omega)\right\}\\ &=Qf(X). \end{aligned}$$

Thus, Qf is invariant with respect to $SO(2)$ and is quasiconvex but not polyconvex. ■

In the theory of hyperelastic materials the total stored energy, $\int_{\Omega} W(\nabla u)\,dx$, is to be minimized. Here, $u:\Omega\to\mathbb{R}^n$ is the deformation of the material and is assumed locally invertible, $\det\nabla u(x)>0 \ \ \forall\ x\in\overline{\Omega}$. The stored energy function W is assumed to be objective, i.e., $W(QF)=W(F) \ \ \forall\ Q\in SO(n)$ and for all $F\in\mathbb{R}_{+}^{n\times n}$ ($=$ the set of $n\times n$ matrices with $\det>0$). Often W is required to be isotropic, i.e., $W\left(QFQ^{\mathsf{T}}\right)=W(F)$ $\forall\ F,Q$ as above. Here, $f=W$ and we do not restrict to $\mathbb{R}_{+}^{n\times n}$ but the notions of quasiconvexity, polyconvexity and rank-one convexity still apply. Thus Theorem 3 provides examples of objective and isotropic quasiconvex functions that are not polyconvex.

Chapter VI

Examples

Construction of non-trivial examples of functions satisfying the hypotheses in the previous theorems is a non-trivial exercise. We let $\Omega \subset \mathbb{R}^2$ be the unit ball, $1 < p < \infty$ for the hypotheses in Theorem 1 and $1 < p < 2$ for the hypotheses in theorems 2 and 3, $f : M^{2\times 2} \to \mathbb{R},\ \ f$ be continuous and satisfy $\left|\boldsymbol{C}\left[X\right]\right|^p \leq f\left(X\right) \leq c_1 + c_2\left|X\right|^p \ \ \forall\ X \in M^{2\times 2}$ where $c_1, c_2 \in \mathbb{R}^+$ and $\boldsymbol{C}$ is a strongly elliptic 4-tensor in all of the following examples.

<u>Example 1</u>

One example of $f\left(X\right)$ that is in our class of functions is Sverak's [11] example where $\boldsymbol{C}$ is the Lamé tensor, $\boldsymbol{C}\left[X\right] = \mu\left(X + X^{\mathsf{T}}\right) + \lambda\left(trX\right)I$, μ and λ are Lamé constants with the bulk modulus, $\mu = \frac{1}{2} = -\lambda$. It is known that $\boldsymbol{C}$ is elliptic if and only if $\mu > 0$ and $2\mu + \lambda > 0$. This problem occurs in linear elasticity and $\boldsymbol{C}\left[X\right]$ represents the stress on a material and $\nabla\phi\left(x\right)$ is the deformation gradient. This gives $\boldsymbol{C}\left[X\right] = \frac{1}{2}\left(X + X^{\mathsf{T}}\right) - \frac{1}{2}\left(trX\right)I = \frac{1}{2}\begin{pmatrix} x_{11} - x_{22} & x_{12} + x_{21} \\ x_{12} + x_{21} & x_{22} - x_{11} \end{pmatrix}$. In his example, $f : M^{2\times 2} \to \mathbb{R}$, $1 < p < 2$, L is a 2-dimensional affine subset of $M^{2\times 2}$ which does not contain any rank-one direction, i.e., $rank\left(A - B\right) \geq 2$ for any 2 distinct $A, B \in L$, and $K \subset L$ is a closed set. In his work, $\left|\ \cdot\ \right|$ is the Euclidean norm and $f\left(X\right) = \left(dist\left(X, K\right)\right)^p$, where $dist\left(X, K\right) = \inf_{Y \in K}\left|X - Y\right|$. His choice of f satisfies the hypotheses of our theorems as follows: In Theorem 1, $f(X)=(dist(X,K))^p$ is clearly continuous. He lets

$L_0=\left\{\begin{pmatrix} s & -t \\ t & s \end{pmatrix},\ s,t\in\mathbb{R}\right\}$ which implies $dist(X,K)\geq dist(X,L_0)$. It can be easily

shown that $dist(X,L_0)=\left|\boldsymbol{C}[X]\right|=\frac{1}{\sqrt{2}}\sqrt{(x_{11}-x_{22})^2+(x_{12}+x_{21})^2}$. Notice L_0 corresponds

to our $L=\left\{X\in M^{n\times n} \mid \boldsymbol{C}[X]=0\right\}$. This then establishes the left inequality in the

hypothesis for f in the statement of Theorem 1. Sverak does not state an upper bound on

f, however it is easily shown that his f satisfies our upper bound. Fix any $X_0\in K$ and

any $X\in M^{2\times 2}$; then $dist(X,K)\leq\left|X-X_0\right|\leq|X|+|X_0|\leq 2^{1/q}\left(|X|^p+|X_0|^p\right)^{1/p}$ by Triangle

Inequality and Holder's Inequality. Therefore,

$$\begin{aligned} f(X)=(dist(X,K))^p &\leq\left(2^{1/q}\left(|X|^p+|X_0|^p\right)^{1/p}\right)^p \\ &=2^{p/q}\left(|X|^p+|X_0|^p\right) \qquad ; \\ &=2^{p-1}|X_0|^p+2^{p-1}|X|^p \end{aligned}$$

if we let $c_1=2^{p-1}|X_0|^p$ and $c_2=2^{p-1}$, we obtain the desired full inequality,

$\left|\boldsymbol{C}[X]\right|^p\leq f(X)\leq c_1+c_2|X|^p$, or in Sverak's terms,

$(dist(X,L_0))^p\leq(dist(X,K))^p\leq\left(2^{1/q}\left(|X|^p+|X_0|^p\right)^{1/p}\right)^p$. As in Sverak, if K is

invariant, then f and Qf are also. Thus the hypotheses for our theorems are satisfied

and Sverak's results follow from our Theorems 1, 2, and 3.

Example 2

Another example is $f(X) = |\boldsymbol{C}[X]|^p$ where $\boldsymbol{C}[X] = \mu(X + X^{\mathsf{T}}) + \lambda(trX)I$, the general Lamé tensor, $1 < p < \infty$, $\mu > 0$, $2\mu + \lambda > 0$. Here,

$L = \{X \in M^{n\times n} \mid \boldsymbol{C}[X] = 0\} = K = \{X \in M^{n\times n} \mid f(X) = 0\}$. Theorem 1 then follows easily as in Example 1. But $L = K$ is convex and thus Theorem 2 does not apply.

However, we can show the invariance of f with respect to $O(2)$. $\boldsymbol{C}[X]$ is invariant with respect to $O(2)$, i.e.,

$$\begin{aligned}
\boldsymbol{C}\left[RXR^{\mathsf{T}}\right] &= \mu\left(RXR^{\mathsf{T}} + \left(RXR^{\mathsf{T}}\right)^{\mathsf{T}}\right) + \lambda\left(tr\left(RXR^{\mathsf{T}}\right)\right)I \\
&= \mu\left(RXR^{\mathsf{T}} + RX^{\mathsf{T}}R^{\mathsf{T}}\right) + \lambda\left(tr\left(RXR^{\mathsf{T}}\right)\right)I \\
&= \mu\left(R\left(X + X^{\mathsf{T}}\right)R^{\mathsf{T}}\right) + \lambda\left(R(trX)R^{\mathsf{T}}\right)I \quad . \\
&= R\left(\mu\left(X + X^{\mathsf{T}}\right) + \lambda(trX)I\right)R^{\mathsf{T}} \\
&= R\boldsymbol{C}[X]R^{\mathsf{T}}
\end{aligned}$$

But $|\boldsymbol{C}[X]|^p$ is convex (see proof of Theorem 1), which implies that $f(X)$ is convex.

Example 3

In this example we let $\boldsymbol{C}[X] = \mu\left(X + X^{\mathsf{T}} - (trX)I\right)$, $L = L_0$ as in Example 1, $K \subset L$, K closed, nonconvex, and $f(X) = g(d)$ where $d = dist(X, K)$, g is continuous and satisfies $c_3 d^p \le g(d) \le c_1 + c_2 d^p$, $c_1, c_2, c_3 \in \mathbb{R}^+$, $1 < p < 2$. We further assume that $g(0) = 0$; then K is as in Theorem 1. To show this choice of f satisfies the hypotheses of Theorem 1, we need $|\boldsymbol{C}[X]|^p \le c_3 d^p$. As in Example 1,

$\left|\boldsymbol{C}[X]\right|^p = \left(\sqrt{2}\mu\sqrt{(x_{11}-x_{22})^2+(x_{12}+x_{21})^2}\right)^p$. The above inequality is true if

$\mu = \frac{1}{\sqrt{2}}(c_3)^{1/p}$ or $c_3 = \left(\sqrt{2}\mu\right)^p$. The growth estimate, $f(X) \le c_1 + c_2 |X|^p$, is as in Example 1 and Theorems 1, 2, and 3 apply. Note that f is invariant if K is invariant.

Example 4

Here we let $g:[0,\infty)\to\mathbb{R}$, be continuous, invariant, and satisfy $c_3 y^p \le g(y) \le c_1 + c_2 y^p \;\; \forall\; y\in[0,\infty)$ where $c_1, c_2, c_3 \in \mathbb{R}^+$, $1 < p < \infty$ for Theorem 1 hypotheses and $1 < p < 2$ for Theorems 2 and 3. Also, $g(0)=0$. Let $K \subset L_0$, K closed and nonconvex with L_0 and $\boldsymbol{C}$ as in Example 1. We give two versions of $f(X)$, Version A and Version B.

Version A

We write $M^{2\times2} = L_0 \oplus N$ where N is a 2-dimensional subspace of $M^{2\times2}$ and $P: M^{2\times2} \to N$ is the projection operator along L_0, thus $\operatorname{Ker} P = L_0$. Let $h: M^{2\times2}\to\mathbb{R}$ be continuous, $h \ge 0$ on $M^{2\times2}$, $K = h^{-1}(\mathbf{0})$, and $c_4\left|P[X]\right| \le h(X) \le c_5 + c_6|X|$, $\forall\; X \in M^{2\times2}$ where $c_4, c_5, c_6 \in \mathbb{R}^+$. Let $f(X) = g(h(X)) \;\; \forall\; X \in M^{2\times2}$.

Now we verify the hypotheses of Theorem 1: by Holder's Inequality,

$$
\begin{aligned}
f(X) = g(h(X)) \le c_1 + c_2 (h(X))^p \\
&\le c_1 + c_2 (c_5 + c_6 |X|)^p \\
&\le c_1 + c_2 \left((c_5^q + c_6^q)^{1/q} (1^p + |X|^p)^{1/p} \right)^p \\
&= c_1 + c_2 (c_7)^{p/q} (1 + |X|^p) \\
&= c_1 + c_2 (c_8 (1 + |X|^p)) \\
&= c_9 + c_{10} |X|^p, \quad c_9 = c_1 + c_2 c_8, \; c_{10} = c_2 c_8.
\end{aligned}
$$

Thus the upper bound on f is satisfied. Now,

$f(X) = g(h(X)) \ge c_3 (h(X))^p \ge c_3 (c_4 |P[X]|)^p \ge c_{11} |\boldsymbol{C}[X]|^p$ by utilizing the following facts:

i) $|\boldsymbol{C}[Y]| \le c_{12} |Y| \quad \forall\, Y \in M^{2\times 2}$,

ii) $\boldsymbol{C}[X] = \boldsymbol{C}[P[X]] \quad \forall\, X \in M^{2\times 2} \quad \forall\, X \in M^{2\times 2}$.

Property i) follows from $\boldsymbol{C}[Y] = \frac{1}{2}\begin{pmatrix} y_{11} - y_{22} & y_{12} + y_{21} \\ y_{12} + y_{21} & y_{22} - y_{11} \end{pmatrix}$ and property ii) follows from writing $X \in M^{2\times 2}$ in the form $X = (I - P)[X] + P[X]$, so $(I - P)[X] \in L_0$. Therefore,

$\boldsymbol{C}[X] = \boldsymbol{C}[(I-P)[X] + P[X]] = \boldsymbol{C}[(I-P)[X]] + \boldsymbol{C}[P[X]] = \boldsymbol{C}[P[X]]$.

Note that $f^{-1}(\mathbf{0}) = K = h^{-1}(\mathbf{0})$ is nonconvex, therefore Theorem 2 applies. Similarly, Theorem 3 applies if K is invariant, and therefore f is also invariant. We also note that $h(X) = dist(X, K)$ satisfies all of the above hypotheses on h and then Sverak's example is a special case of Version A where $g(y) = y^p$.

Version B

Let $h : M^{2\times 2} \to \mathbb{R}$ be continuous, $h \geq 0$ on $M^{2\times 2}$, $K = h^{-1}(\mathbf{0})$, and $c_{11} dist(X, K) \leq h(X) \leq c_5 + c_6 |X|$, $\forall X \in M^{2\times 2}$ where $c_{11}, c_5, c_6 \in \mathbb{R}^+$. Let $f(X) = g(h(X))$ $\forall X \in M^{2\times 2}$. Verification of the upper bound on f is the same as in Version A. For the lower bound we have

$$\begin{aligned} f(X) = g(h(X)) &\geq c_3 (h(X))^p \\ &\geq c_3 (c_{11} dist(X, K))^p \\ &= c_3 (c_{11})^p (dist(X, K))^p \\ &\geq c_{12} (dist(X, L_0))^p, \quad \text{where } c_{12} = c_3 (c_{11})^p \\ &= c_{12} |\boldsymbol{C}[X]|^p. \end{aligned}$$

From this the three Theorems apply.

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List of References

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Vita

Catherine S. Remus was born in Great Lakes Naval Hospital, Illinois on October 29, 1952. She lived with her family in Illinois and attended Parochial school through tenth grade. She graduated from Willowbrook High School in June of 1970. She attended various colleges from 1970 until 1976 because of family relocations. She worked in Civil Engineering from 1972 until 1988 in increasingly responsible positions. She was married in 1984 and gave birth to her son in 1988. She focused on family responsibilities until 1997 when she divorced and went back to school. She earned a Bachelor of Science in Mathematics in May, 1999 from the University of West Florida and began her graduate studies at the University of Tennessee in August, 1999 where she earned a Master of Science in Mathematics in December, 2003.