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A Holographic model of Striped Superconductors

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I am submitting herewith a dissertation written by Suman Ganguli entitled "A Holographic model of Striped Superconductors." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Physics.

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A Holographic model of Striped Superconductors

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Abstract

One of the most prominent distinguishing features in strongly correlated electron systems, such as the high Tc (critical temperature) cuprates and the most recent iron pnictides, is the presence of "competing orders" that are related to the breaking of the lattice symmetries. Does the ubiquitous presence of such inhomogeneous orders in strongly correlated superconductors have a deep connection to superconductivity? The answer to this question is crucial for identifying the mechanism of superconductivity, at least in the cuprates. Amidst serious difficulties within conventional theoretical framework to deal with strongly interacting degrees of freedom at finite density, "AdS/CFT correspondence" or "gauge/gravity duality" sheds new light into the origin of high Tc superconductivity by mapping the original system into an appropriate weakly coupled one. An example of the "Holographic principle", according to which, a quantum theory with gravity must be describable by a boundary theory, AdS/CFT duality provides guidelines to model a d dimensional strongly coupled condensed matter system in terms of a suitable gravity theory (as low energy limit of String theory) on a d+1 dimensional anti de-Sitter (AdS) space. In this thesis I will develop a phenomenological holographic model of strongly coupled "striped" superconductors in two spatial dimensions and study the interplay between charge density waves and superconductivity. It will be shown that charge density waves with large modulation compete with superconducting order, causing the critical temperature to fall off with increasing modulation in various ways depending on free parameters of the theory. For small modulation, the effects of fluctuations dominate,

causing an enhancement of critical temperature upon turning on modulation. The highest critical temperature is obtained at an intermediate modulation. Moreover, there exists a region in parameter space of the theory within which the modulation vs Tc phase structures show striking resemblance to doping phase structure of cuprates.

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Chapter 1

Introduction

It was noticed in the early part of the 20^{th} century that the electrical resistivity of most metals drops suddenly to zero as the temperature is lowered below a critical temperature T_c. These materials were called superconductors. A second independent property of these materials was the Meissner effect: A magnetic field is expelled when $T < T_c$. This is perfect diamagnetism and does not follow from the perfect conductivity (which alone would imply that a pre-existing magnetic field is trapped inside the sample).

A phenomenological description of both of these properties was first given by the London brothers in 1935 with the simple equation $J_i \propto A_i$ [1]. Taking a time derivative yields $E_i \propto \partial J_i/\partial t$, showing that electric fields accelerate superconducting electrons rather than keeping their velocity constant as in Ohm's law with finite conductivity. Taking the curl of both sides and combining with Maxwell's equations yields $\nabla^2 B_i \propto B_i$ showing the decay of magnetic fields inside a superconductor.

In 1950, Landau and Ginzburg described superconductivity in terms of a second order phase transition whose order parameter is a complex scalar field ψ [2]. The density of superconducting electrons is given by $n_s = |\psi(x)|^2$. The contribution of ψ

to the free energy is assumed to take the form

$$F = \alpha(T - T_c)|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \cdots \tag{1.1}$$

where α and β are positive constants and the dots denote gradient terms and higher powers of φ. Clearly for $T > T_c$ the minimum of the free energy is at $\psi = 0$ while for $T < T_c$ the minimum is at a nonzero value of ψ. This is just like the Higgs mechanism in particle physics, and is associated with breaking a $U(1)$ symmetry. The London equation follows from this spontaneous symmetry breaking [4].

A more complete theory of superconductivity was given by Bardeen, Cooper and Schrieffer in 1957 and is known as BCS theory [5]. They showed that interactions with phonon can cause pairs of elections with opposite spin to bind and form a charged bound state called a Cooper pair. Below a critical temperature T_c, there is a second order phase transition and these bosonic pairs condense. The DC conductivity becomes infinite producing a superconductor. The pairs are are not bound very tightly and typically have a size which is much larger than the lattice spacing. In the superconducting ground state, there is an energy gap Δ for charged excitations. This gap is typically related to the critical temperature by $\Delta \approx 1.7T_c$. The charged excitations are "dressed electrons" called quasi particles. The gap in the spectrum results in a gap in the (frequency dependent) optical conductivity. If a photon of frequency ω hits the superconductor, it must produce two quasi particles. The binding energy of the Cooper pair is very small, but the energy of each quasi particle is Δ, so the gap in the optical conductivity is $\omega_g = 2\Delta \approx 3.5T_c$.

A new class of high T_c superconductors were discovered in 1986 [6]. They are cuprates and the superconductivity is along the CuO_2 planes. The highest T_c known today (at atmospheric pressure) is $T_c = 134^o K$ for a mercury, barium, copper oxide compound. If you apply pressure, T_c climbs to about 160K. There is evidence that electron pairs still form in these high T_c materials, but the pairing mechanism is not well understood.

Another class of superconductors were discovered in 2008 based on iron and not copper [7]. The highest T_c so far is $56^o K$. These materials are also layered and the superconductivity is again associated with the two dimensional planes. They are called iron pnictides since they involve other elements like arsenic in the nitrogen group of the periodic table.

Bolstered by numerous experiments, strongly coupled systems are expected to play a key role in the understanding of high temperature superconductivity. However, strongly coupled systems are far from being understood [12, 13]. Study of such systems within conventional theoretical frameworks have met with serious difficulties. While some techniques have been devised over the years to model such systems, there is a scarcity of simple, tractable models to deal with strongly coupled fermions at finite density. Recently, there has been a flurry of activity trying to fill in this gap using various developments of the gauge/gravity duality [14]. Holographic ideas have become increasingly popular in the field of high energy theory, over last fifteen years. The AdS/CFT correspondence, spurred by proposals in references [8, 9, 10], has become the most successful realization of holographic principle. The conjecture posits a correspondence which relates string theory on asymptotically anti de Sitter (AdS) space-time to conformal field theory on the boundary. The first and most fundamental reason for its importance is that, AdS/CFT conjecture provides a definition of quantum gravity in a particular curved background space-time. The second is that, AdS/CFT acts as a tool to understand strongly interacting field theories with Lorentz symmetry in d space-time dimensions by mapping them to classical gravity, as low energy limit of string theory, in $d + 1$ space-time dimensions. A popular model that includes some important ingredients of a realistic system is the 2+1 dimensional holographic superconductor [16, 15]. It consists of a 3+1-dimensional Einstein-Maxwell-scalar theory in an AdS black hole background. At low temperatures, a phase transition to a superfluid state takes place [11], in which the scalar field develops a non-vanishing expectation value, spontaneously breaking the gauge $U(1)$ symmetry. Physical properties such as transport coefficients can be studied and contrasted with other known systems like Bardeen-Cooper-Schrieffer (BCS) superconductors. The main conceptual ingredients of the holographic superconductors are strong coupling and proximity to an underlying conformal symmetry, which are believed to be crucial features of cuprates as well.

One of the differences between conventional superconductors and high temperature superconductors is that the normal states of the conventional superconductors are well described by Fermi liquid, whose only (weak coupling) instability is to superconductivity. By contrast, the normal states of high temperature superconductors, such as cuprates and iron pnictides, are highly correlated and thus, exhibit other low temperature orders which interact strongly with superconductivity. One of the prominent orders is the unidirectional charge density wave "stripe" order [56, 55, 57] that break the discrete translation and rotation symmetries of the square lattice underlying the CuO_2 planes. It is therefore important to understand the nature of the interplay between superconductivity and the stripe order in the presence of strong correlation. Models have been proposed based on the coexistence of homogeneous superconductivity with CDW and SDW, as well as models where the superconducting order parameter itself is modulated (pair density waves or PDW). Signatures of CDW have been reported in a variety of strongly correlated superconductors, most notably the hole-doped cuprates $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$, $La_{1.8-x}Eu_{0.2}Sr_xCuO_4$, and $La_{2-x}Ba_xCuO_4$ [17]. For these materials, the order can be consistently interpreted in terms of uni-directional SDW and CDW over a wide range of doping. Study of these orders and their connection to superconductivity required extension of existing holographic models of homogeneous systems by introducing inhomogeneity.

In this thesis, we will develop a holographic model to study the interplay between superconductivity and CDW orders. The outline of this thesis is as follows. Chapter two and chapter three will be dedicated to discussion of superconductivity and holography respectively. In chapter four we will review a holographic model of homogeneous superconductors. In chapter five, we will introduce the system describing strongly coupled striped superconductors and solve the system in both mean-field level and then adding fluctuations. The last chapter will summarize the main results and conclusions of this work.

Chapter 2

Superconductivity

Quantum theories have successfully explained a variety of fascinating phenomena throughout the last century. These phenomena can be loosely classified as either microscopic or macroscopic quantum phenomena. The former class of phenomena are primarily observed at small length scales or at high energies, while the later are observed at low temperatures in systems with large degrees of freedom. Superconductivity, superfluidity and Bose-Einstein condensation, are all examples of macroscopic quantum phenomena. These macroscopic systems undergo phase transitions between ordered and disordered states, under variation of external conditions. A transition from a disordered to an ordered state is characterised by non-vanishing expectation value of some suitable macroscopic coherent state, called order parameter, wherein a macroscopic number of elementary excitations condense into a ground state that exhibits long ranged order among spatially separated excitations.

Superconductors exhibit fascinating phenomena that can be predicted, at least for conventional superconductors, with great accuracy. To explain these phenomena both macroscopic and microscopic models are used. There are macroscopic models like that of Ginzburg and Landau, in which cooperative states of electrons are represented by a complex scalar field. There are microscopic models like BCS theory, in which electrons appear explicitly, but are assumed to interact via only single phonon exchange. These approximations are not useful for high T_c phenomena where electrons are strongly correlated. So it is necessary to sort out the general properties that are independent of these models. There is one feature common to all these models that provide these high-precision predictions : all these models exhibit a spontaneous breakdown of electromagnetic gauge invariance in a superconductor. The thermodynamic conditions and underlying mechanism of this symmetry breaking depends on the details dynamical models. However, many fundamental characteristics of superconductors can be predicted by simply assuming that for whatever reason the electromagnetic gauge invariance of the system is spontaneously broken [4].

In the first section of this chapter we will glance through two popular theories of weakly coupled superconductors: Ginzburg-Landau theory and BCS theory *. In the following section, after a brief review of electromagnetic gauge invariance and spontaneous symmetry breaking, we will discuss how such a simple consideration can account for much of the phenomonology of superconductivity.

2.1 The Ginzburg-Landau and BCS theory

The Ginzburg-Landau (GL) theory of superconductivity [2], originally introduced as a phenomological theory, describes superconducting phase transition from a thermodynamics point of view. In orginal proposal a wave function $\psi(x)$ was intoduced as a complex order parameter. The quantity $|\psi|^2$ is to represent the local density of superconducting electrons $n_s(x)$. The theory was developed by applying variational method to an assumed expansion of free energy in powers of $|\psi|^2$ and

*In 1959, Gorkov derived [3] the macroscopic Ginzburg-Landau theory from microscopic BCS theory.

$|\nabla\psi|^2$, leading to a pair of coupled differential equations for $\psi(x)$ and electromagnetic potential $A(x)$.

The basic postulate of GL theory is that if ψ is small and varies slowly in space, the free energy density can be expanded as,

$$f = f_n + \alpha(T)\,|\psi|^2 + \frac{\beta(T)}{2}\,|\psi|^4 + \frac{1}{2m}|D\psi|^2 \tag{2.1}$$

where,

$$f_n = f_{n0} + \frac{B^2}{8\pi}$$

is the free energy of the normal state; $\alpha(T)$ and $\beta(T)$ are temperature dependent phenomological paramaters of the theory and $D = \nabla - i\,q\,\vec{A}$ is the spatial part of gauge derivative.

Here we consider a spatially uniform superconductor without any external magetic field ($D\psi = 0$). The difference between free energies in normal and superconducting states reads,

$$f_s - f_n = \alpha(T)\,|\psi|^2 + \frac{\beta(T)}{2}\,|\psi|^4 = |\psi|^2\left(\alpha(T)\,+\frac{\beta(T)}{2}\,|\psi|^2\right) \tag{2.2}$$

Note that β must be positive for free energy to be bounded below (or else we have to include highier power in expansion). The above functional is minimized by,

$$|\psi|^2 = 0, -\frac{\alpha}{\beta} \tag{2.3}$$

If sign of parameter α determines which of the above two solutions represents the ground state. If $\alpha > 0$ then the minimum free energy occurs at $|\psi|^2 = 0$ and if $\alpha < 0$ then the minimum free energy occurs at $|\psi|^2 = -\frac{\alpha}{\beta}$.

Since by definition T_c is the highest temperature at which a non vanishing condensate gives a lower free energy than a vanishing condensate, the parameter $\alpha(T)$ must change sign at T_c, going from positive above T_c to negative below T_c. We also assume that $\beta(T)$ is a smooth function of temperature near T_c. We can thus expand these parameters about T_c as,

$$\alpha(T) = \dot{a}(T - T_c) + \cdots \quad ; \quad \beta(T) = b + \cdots \tag{2.4}$$

Then close to T_c the magnitude of condensate becomes,

$$|\psi| = \sqrt{\frac{\dot{a}}{b}}\,(T_c - T)^{1/2} \qquad\qquad T < T_c$$

Inserting the above expressions for $|\psi|$, α and β the difference between free enrgy densities between superconducting phase and normal phase reads,

$$f_s - f_n = -\frac{\dot{a}^2}{2b}\,(T - T_c)^2 \tag{2.5}$$

We note that the entropy density ($s = -\partial f/\partial T$), is continuous during the transition.

$$s_s(T) - s_n(T) = -\frac{\dot{a}^2}{b}(T_c - T)$$

However, the heat capacity per unit volume $C_V = -T\,\partial^2 f\,/\,\partial T$ has a discontinuity at T_c.

$$\Delta\, C_V = T_c \frac{\dot{a}^2}{b} \tag{2.6}$$

Since the system exhibits discontinuity in a second derivative of the free energy, it is a second order phase transition.

The Ginzburg-Landau theory makes many important predictions, a description of which can be found in any standard text book on Condensed matter physics [21].

However, it is worth mentioning here two of the most powerful features of this theory: first, this theory, although originally applied as mean-field theory can be extended to include thermal fluctuations, an important feature of high T_c superconductors. Second, the theory does not rely on underlying microscopic theory and consequently applicable upon full filling certain conditions even if the microscopic mechanism is unknown.

The first microscopic theory of superconductivity was worked out by Bardeen, Cooper and Schrieffer in 1957 [5]. Since it's inception, the BCS theory successfully explained a plethora of experimental data. In particular, explanation of isotope effect and prediction of existence of an energy gap at Fermi surface are two major triumphs of the theory.

The BCS theory was formulated upon three key ideas. First, it was recognised that effective interaction between electrons can be attractive. Second Cooper's simple consideration of just two electrons outside an occupied Fermi surface showed that the electrons form a stable bound state in the presence of attractive potential. Moreover such states form as long as the effective interaction between electrons remain attractive, however weak it is. Third a coherent state N body wave function was constructed by Schrieffer such that all electrons near to the Fermi surface pair up. These bound states called Cooper pairs are defined as,

$$\Phi^\dagger(\vec{R}) \equiv \int d^3r \, \phi(\vec{r}) \, \psi^\dagger_\uparrow(\vec{R} + \vec{r}/2) \, \psi^\dagger_\downarrow(\vec{R} - \vec{r}/2) \tag{2.7}$$

where $\psi^\dagger$ are Fermionic operators and $\phi(\vec{r})$ is the spatial part of two body wave function of spin singlet state. The effective interaction between electrons due to exchange of a virtual phonon of wave is of the form:

$$V_{eff}(Q, \omega) = |g_Q|^2 \frac{1}{\omega^2 - \omega_Q^2} \tag{2.8}$$

where g_Q is the electron-phonon vertex. It was shown by Migdal that the electron-phonon vertex,

$$g_Q \approx \sqrt{\frac{m}{M}} \tag{2.9}$$

where m is the effective mass of electrons at Fermi surface and M is the mass of ions. To simplify the problem individual interaction vertex between a phonon mode with wavevector Q and electrons are replaced by a contant avarage $|g_{eff}|$ and individual phonon fequencies are replaced by Debye frequency ω_D. Then the effective potential reads,

$$V_{eff}(Q,\omega) = |g_{eff}|^2 \frac{1}{\omega^2 - \omega_D^2} \tag{2.10}$$

A fundamental parameter of BCS theory is the electron-phonon coupling parameter λ defined as,

$$\lambda = |q_{eff}|^2 \, g(\epsilon_F) \tag{2.11}$$

and is assumed to be small, $\lambda << 1$. Consequently most applications of this theory is valid only in this the weak coupling regime. This approximation breaks down for high T_c superconductors and the theory become strongly coupled.

2.2 Superconductivity via Higgs Mechanism

Although, breaking of gauge symmetry first appeared in Ginzburg-Landau theory [2] in 1950, the mechanism was not clear at that time. In 1962, Schwinger [22] proposed what we know today as breaking of gauge symmetries in relativistic theories. The following year, Anderson [23] identified Ginzburg-Landau theory as a non-relativistic realization of Schwinger's idea and thereby established superconductivity as a theory of spontaneous breaking of gauge symmetries. In 1964, Higgs [25], Englert and Brout [26] introduced relativistic models of electroweak symmetry breaking analogous to

Ginzburg-Landau model of superconductivity. In 1981, Littlewood and Varma [28] realized that an unexpected feature of the Raman spectrum of $NbSe_2$ superconductor could be explained by a massive collective mode the oscillation of the amplitude of the superconducting gap.

2.2.1 Spontaneous Symmetry Breaking

A spontaneously broken symmetry in field theories is associated with a degeneracy of vacuum states. Consider an action $I[\psi(x)]$ of a single field $\psi(x)$. The action has a symmetry under a symmetry transformation $g : \psi(x) \to \psi'(x)$ if,

$$I[g\,\psi] = I[\psi] \tag{2.12}$$

The vacuum is defined as a state whose expectation values of $\psi(x)$ is at a minimum of vacuum energy $-I[\psi]$, say at ψ_0. Since the action is invariant under a symmetry transformation g, under such a transformation the same minimum of vacuum energy $-I[g\psi] = -I[\psi]$ is obtained, but now at $g\psi_0$. So, unless $g\psi_0 = \psi_0$, we have two minima, each corresponding to a state of broken symmetry. However, it should be emphasized that Spontaneous symmetry breaking occurs only for idealized systems, that are infinitely large. The matrix elements of Hamiltonian between vacuum states of different field expectation values are exponentially suppressed by the size of the system and vanish for infinitely large systems. Hence, the true vacuum is one of the above minima and not any linear combination of them (which may also preserve the symmetry of the action).

In order to describe symmetry breaking in physical systems, we may introduce a quantity, different values (or sets of values) of which correspond to different symmetry structure. In descriptions of a phase transition characteristic values of such a quantity determines if the system is in an "ordered" or a "disordered" phase. For this reason such quantities are called *order parameter*. In what follows we will consider the set of values of an order parameter as points in an infinitely differentiable manifold M. Let G be a compact Lie group, acting smoothly on M. The action of a group element, $g \in \mathrm{G}$, on a point $\psi \in M$ will be denoted simply as $g\psi$. For each point we also denote as H_ψ the set of all group elements that leave ψ invariant, that is, $\mathrm{H}_\psi = \{g \in \mathrm{G} \,|\, g\psi = \psi\}$. This set forms a subgroup of G and is called the *little (or isotropy) group* of ψ. In physical terms, it consists exactly of those transformations that left unbroken when the order parameter takes the value ψ. A very important notion that we will further use is that of the *orbit* $\mathcal{G}(\psi)$ of a given point ψ. It consists of all points of M which can be reached from ψ by a (naturally, broken) symmetry transformation, $\mathcal{G}(\psi) = \{g\psi \,|\, g \in \mathrm{G}\}$. The relation defined by the condition that two points be connected by a group of transformations is an equivalence relation, and the group orbits then define a partition of the manifold M into equivalence classes. Any potential V on the manifold which is invariant under the group action, $V(\psi) = V(g\psi)$ for all $\psi \in M$ and $g \in \mathrm{G}$, may be thought of as a function on the orbits. The minimization problem for a given potential on M can therefore be reformulated as a minimization of a function on the space of orbits.

It is clear from the definition of the orbit that two points ψ and ψ' on the same orbit have isomorphic little groups, since for each symmetry transformation $h \in \mathrm{H}_\psi$ we can define another symmetry transformation $h' = g\,h\,g^{-1}$ where $\psi' = g\psi$ so that $h'\psi' = \psi'$. This is in fact an immediate consequence of the stronger statement that the two little groups are *conjugate*, $\mathrm{H}_{g\psi} = g\,\mathrm{H}_\psi\,g^{-1}$. In addition to the orbit $\mathcal{G}(\psi)$ there may be other points on the manifold whose little groups are conjugate to H_ψ. For example, when M is a linear space , multiplying ψ by any (nonzero) number we obviously get a point with the same little group as ψ. The set of all points with little groups conjugate to H_ψ is called a *stratum*, $S(\psi)$. Intuitively, a stratum consists of all points of the same symmetry "*class*": they have the same unbroken subgroup and the same symmetry-breaking pattern. In the phase diagram, a stratum would be associated with a particular phase. At a phase transition, the order parameter moves from one stratum to another, and the symmetry class changes.

One of the most striking consequences of spontaneous symmetry breaking is the existence of soft modes in the spectrum whose energy vanishes in the long-wavelength limit. This is the *Goldstone theorem* and the soft modes are usually referred to as the *Nambu-Goldstone* bosons. Any system described by a Lagrangian with symmetry group G, when in a phase in which G is spontaneously broken to a subgroup H, will posses a set of fields, that transform under G like the coordinates of coset space G/H. Such fields are called Nambu-Goldstone excitations. From a physical point of view, the most important ingredient responsible for the presence of NG bosons is, apart from the symmetry breaking itself, the existence of a conserved charge. The Goldstone theorem guarantees the existence of a NG mode in the spectrum. In the most general formulation it does not tell us how many NG bosons there are. In Lorentz invariant theories there turns out to be exactly one NG boson for each broken generator. In spontaneously broken global symmetries these bosons are massless and spin zero particles. In spontaneously broken local symmetries these degrees of freedom show up as helicity zero states of vector particles associated with the broken local symmetries. Consequently each vector particle acquires a mass. This phenomenon is known as *Higg's mechanism.* Several consequences of broken symmetries can be deduced solely from properties of these Nambu-Goldstone modes.

2.2.2 Electrodynamics and Gauge Invariance

The need for a principle of gauge invariance arises from the difficulty in formulating quantum theories of massless particles with spin. It turns out that there is no way to construct a true four vector as a linear combination of creation and annihilation operators for massless particles with helicity ± 1. We can construct an antisymmetric tensor of form $F_{ab} = \partial_a A_b - \partial_b A_a$, but A_a's transform as four-vector only up to a guage transformation,

$$U(\Lambda)\, A_a\, U(\Lambda) = \Lambda_a^b\, A_b(\Lambda x) + \partial\, \Omega(x, \Lambda) \tag{2.13}$$

where $U(\Lambda)$ is a unitary representation of Lorentz transformation Λ. The presence of singularities at $m = 0$ in propagator of a massive vector field of helicity ± 1 prevents us from passing to $m \to 0$ limit from the theory of massive particle of spin one.

If all interactions take place through only F_{ab} and its derivatives, these problems could be avoided, as due to commutativity of partial derivative, field strength tensor F_{ab} is blind to gauge freedom. However this is neither the most general interaction, nor the one realized in nature. To remove this restriction, and incorporate A_a itself as directly interacting with matter. then matter action I_M must be invariant under general guage transformation,

$$A_a(x) \to A_a(x) + \partial_a \epsilon(x) \tag{2.14}$$

The change in matter action under the transformation 2.14,

$$\delta I_M = \int d^4x\, \frac{\delta I_M}{\delta A_a(x)} \delta A_a(x) = \int d^4x\, \frac{\delta I_M}{\delta A_a(x)} \partial_a \epsilon(x) \tag{2.15}$$

Integrating by parts (and thus shifting the ∂_a to $\frac{\delta I_M}{\delta A_a(x)}$) with vanishing boundary term, leads to the condition for which change in matter action will vanish,

$$\partial_a \frac{\delta I_M}{\delta A_a(x)} = 0 \tag{2.16}$$

This condition is satisfied trivially if I_M involves only F_{ab} and its derivatives, along with other matter fields, but is non-trivial if I_M involves A_a itself.

Suppose in the set of independent fields other than A_a , fields are denoted by $\psi^{(l)}(x)$, where l runs over number of independent fields. We want this set of fields to provide conserved currents when coupled to $A_a(x)$. We know that, infinitesimal internal symmetries of the action imply the existence of conserved currents. Under an infinitesimal transformation $\epsilon(x)$, these fields changes by,

$$\delta\psi^{(l)}(x) = i\epsilon(x)\, q_l\, \psi^{(l)}(x) \tag{2.17}$$

If the above transformations leave the matter action invariant for a constant ϵ then for general $\epsilon(x)$ the change in matter action must take the form,

$$\delta I_M = -\int d^4x J^a(x)\partial_a\epsilon(x) \tag{2.18}$$

When the matter fields satisfy their equations of motions, the matter action is stationary with respect to any variation of the $\psi^{(l)}$. So in this case,

$$\partial_a J^a = 0 \tag{2.19}$$

where,

$$J^a = -i\sum_l \frac{\partial\mathcal{L}_M}{\partial(\partial_a\psi^{(l)})}\, q_l\, \psi^{(l)} \tag{2.20}$$

The time independent charge operator reads,

$$Q = \int d^3x\, J^0 \qquad ; \quad [Q, \psi^{(l)}(x)] = -q_l\, \psi^{(l)}(x) \tag{2.21}$$

We can therefore construct a Lorentz invariant theory by coupling the vector field A_a to the conserved current J^a, in the sense that $\delta I_M/\delta A_a(x)$ is taken to be proportional to $J^a(x)$. Any constant of proportionality may be absorbed into the definition of overall scale of charges q_l. So,

$$\frac{\delta I_M}{\delta A_a(x)} = J^a(x) \tag{2.22}$$

This requirement can be restated as a *Principle of Invariance* : The matter action is invariant under the joint transformations,

$$\delta A_a(x) = \partial_a \epsilon(x) \tag{2.23}$$

$$\delta\psi^{(l)}(x) = i\,\epsilon(x)\,q_l\,\psi^{(l)}(x) \tag{2.24}$$

A symmetry of this type with an arbitrary $\epsilon(x)$ is called a *Local Symmetry* or *Gauge Invariance of Second kind.* A symmetry under a transformation with ϵ a constant, is called a *Global Symmetry* or a *Gauge Invariance of First kind.*

The action for photons themselves takes the form,

$$I_\gamma = -\frac{1}{4}\int d^4x F_{ab}F^{ab} \tag{2.25}$$

The field equations for electromagnetism reads,

$$0 = \frac{\delta}{\delta A_b}[I_\gamma + I_M] = \partial_a F^{ab} + J^b \tag{2.26}$$

We note that the change in F_{ab} under local gauge transformation 2.14 ,

$$\delta F_{ab} = \partial_a(\delta A_b) + \partial_b(\delta A_a) = \partial_a\partial_b\epsilon(x) + \partial_b\partial_a\epsilon(x) = 0 \tag{2.27}$$

A finite gauge transformation can be obtained by simply exponentiating infinitesimal transformation. The gauge group can be parametrised by a function $\Omega(x)$ so that it acts on gauge field and matter field as,

$$A_a(x) \rightarrow A_a(x) + \partial_a \Omega(x) \tag{2.28}$$

$$\psi^{(l)}(x) \rightarrow e^{i\, q_l\, \Omega(x)} \psi^{(l)}(x) \tag{2.29}$$

Given a gauge field $A_a(x)$, in general it is not possible to choose $\Omega(x)$ such a way that all four components of gauge field vanish in a finite region region.For this the function $\Omega(x)$ has to satisfy four differential equations,

$$A_a(x) + \partial_a \Omega(x) = 0 \tag{2.30}$$

which cannot be solved unless certain integrability conditions are satisfied. However if there exist such a function $\Omega(x)$, for which all components of gauge field vanish, then all elements of F_{ab} vanish identically. A gauge field is called a *pure gauge* field if there exists a gauge transformation which makes it vanish everywhere. Consequently the necessary and sufficient condition that F_{ab} vanish everywhere is that the gauge field must be expressible as a pure gauge field, satisfying (2.30).

2.2.3 Broken Gauge Invariance and Nambu-Goldstone modes

A superconducting system is described by an order parameter ψ charged under electromagnetic gauge group $G = U(1)$. The action for the system is invariant under gauge transformations of the form,

$$A_a(x) \rightarrow A_a(x) + \partial_a \Omega(x) \tag{2.31}$$

$$\psi(x) \rightarrow e^{i\, q\, \Omega(x)} \psi(x) \tag{2.32}$$

where $\Omega(x)$ is arbitrary and $q = 2\,e$ is the charge of the ψ under this $U(1)$ symmetry. Therefore the group is compact: the phases $\Omega(x)$ and $\Omega(x) + \frac{2\pi}{e}$ must be identified. The normal state of a superconductor is characterized by a vanishing order parameter $\psi(x) = 0$. The isotropy group of this value of order parameter is the entire gauge group $H_{\psi=0} = U(1)$. A non-vanishing order parameter ψ of charge $q = 2e$ breaks this $U(1)$ symmetry into a subgroup Z_2 with two elements $\Omega(x) = 0$ and $\Omega(x) = \frac{\pi}{e}$.

In our case there will be a single Nambu-Goldstone excitaion described by a field $\phi(x)$. We write the order parameter $\psi(x)$ as,

$$\psi(x) = e^{i\,q\,\phi(x)}\rho(x) \tag{2.33}$$

where $\rho(x)$ is a gauge invariant field. Under $U(1)$ the Nambu-Goldstone excitation transforms like the phase $\Omega(x)$ itself

$$\phi(x) \rightarrow \phi(x) + \Omega(x) \tag{2.34}$$

so that under a gauge transformation $\psi(x) = e^{i\,q\,\phi(x)}\rho(x) \rightarrow e^{i\,q\,(\phi(x)+\Omega(x))}\rho(x) = e^{i\,q\,\Omega(x)}\psi(x)$. The field $\phi(x)$ parameterizes the coset space $U(1)/Z_2$. So we must identify the points

$$\phi(x) = \phi(x) + \frac{\pi}{e} \tag{2.35}$$

The matter action describing such a system must be some gauge invariant functional of $\rho(x)$, $A_a(x)$ and $\phi(x)$. Since $\rho(x)$ is gauge invariant by definition and first derivatives of ϕ transform the same way as gauge field themselves, the matter action must be a functional of ρ and $\partial_a\phi - A_a$. In what follows, we will assume that the superconductor has a stable equilibrium configuration in the absence of Goldstone or external electromagnetic fields, so that the energy has at least a local minima at

$$\partial_a \phi(x) - A_a(x) = 0 \tag{2.36}$$

with non-vanishing second derivative with respect to $\partial_a \phi(x) - A_a(x)$. In other words, deep inside a large superconductor where boundary conditions are unimportant the electromagnetic field is pure gauge,

$$\partial_a \phi = A_a \tag{2.37}$$

and consequently electromagnetic field vanishes. Such expulsion of an external electromagnetic field, in particular an applied magnetic field by a superconductor is known as *Meissner effect*.

In a simply connected time-independent superconductor the Nambu-Goldstone fields can be gauged away. However, in multiply connected superconductors such a gauge transformation may not be possible, since the $\phi(x)$ can jump by multiples of π/e. Consider a thick superconducting wire with cross sectional diameter much larger than λ, bent into a closed ring. If we draw a closed contour $\mathcal{C}$ running deep inside the wire, then previous analysis shows that $\nabla \phi - \vec{A}$ vanishes along contour. ie,

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{x} = \oint_{\mathcal{C}} \nabla \phi \cdot d\vec{x} = \oint_{\mathcal{C}} d\phi$$

Since going around a ring ϕ must return to an equivalent value and can therefore only change by some integral multiple of π/e, we can write,

$$\oint_{\mathcal{C}} d\phi = \frac{n\pi}{e}$$

Using Stoke's theorem we see that the magnetic flux through the area $\mathcal{A}$ surrounded by contour $\mathcal{C}$ is quantized,

$$\int_{\mathcal{A}} \vec{B} \cdot d\vec{S} = \frac{n\pi}{e} \tag{2.38}$$

This is known as *flux quantization.* The flux quantization shows that the current (flowing through a layer of thickness λ below the surface of superconductor) maintaining the magnetic field cannot decay smoothly, but only in jumps such that the magnetic flux drops by multiples of π/e. So there is no ordinary electrical resistance.

In order to construct a Lagrangian density, we note that the only non-trivial irreducible representation of $U(1)$ is a real two vector (ψ_1, ψ_2) or equivalently a Golstone mode $\phi(x)$ and modulus $\rho(x)$, with

$$\psi_1 + i\,\psi_2 = \rho\, e^{q\,\phi(x)} \equiv \psi(x) \tag{2.39}$$

Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} - (D_a \psi)^\dagger D^a \psi - V(\psi) \tag{2.40}$$

with,

$$V(\psi) = \alpha(T)\, \psi^\dagger \psi + \frac{\beta(T)}{2} (\psi^\dagger \psi)^2 \tag{2.41}$$

where, $D_a = \partial_a - i\,q\,A_a$. In terms of ρ and ϕ the Lagrangian density (2.40) reads,

$$\mathcal{L} = -\frac{1}{4} F^{ab} F_{ab} - \partial_a \rho\, \partial^a \rho - \alpha(T) \rho^2 - \frac{\beta(T)}{2} \rho^4 - q^2\, \rho^2\, (\partial_a \phi - A_a)\, (\partial^a \phi - A^a) \tag{2.42}$$

This is the Ginzburg-Landau theory of superconductivity derived by Gorkov [3] from microscopic BCS theory in the case of short range potential and a temperature close to T_c. The equations of motion read,

$$\partial_a\, F^{ab} - 2q^2 \rho^2 \left(\partial^b \phi - A^b\right) = 0 \tag{2.43}$$

$$\Box\,\rho - \alpha(T)\,\rho - \beta(T)\rho^3 - q^2\,\rho\,(\partial_a\phi - A_a)\,(\partial^a\phi - A^a) = 0 \tag{2.44}$$

$$\Box\,\phi - \frac{A^a}{q^2\,\rho^2} = 0 \tag{2.45}$$

Let's consider a superconductor of volume L^3 in an external magnetic field B. We define a characteristic length depending on nature of material of superconductor,

$$\lambda = \frac{1}{q|\langle\rho\rangle|} \tag{2.46}$$

In static case we have $A_0 = \dot{\phi} = 0$. So the energy contribution of the third term in (2.42) must be of order $\lambda^{-2}\,|\nabla\phi - \vec{A}|^2\,L^3$. If a magnetic field of order B penetrated the superconductor, then we would have $|\nabla\phi - \vec{A}| \sim BL$. So the energy cost of allowing the magnetic field into the superconductor would be of order $\lambda^{-2}B^2L^5$. The energy cost of expelling a magnetic field B from a volume L^3 is of order $B^2\,L^3$. Hence a weak magnetic field will be expelled from a superconductor if $\lambda^{-2}B^2L^5 \gg B^2L^3$ or if $L \gg \lambda$ (*Meissner effect*). For this reason the parameter λ is known as *penetration depth* of superconductor.

Variations in $\rho(x)$ are characterized by a distance scale known as *coherence length*, given by,

$$\xi = \frac{1}{\sqrt{2\,|\alpha|}} \tag{2.47}$$

The difference in energy per unit volume between the normal state and superconducting state is given by,

$$\Delta = -\frac{\alpha^2}{2\beta} \tag{2.48}$$

Eliminating α and β we obtain an approximate relation between observables,

$$\Delta = \frac{1}{8e^2\lambda^2\xi^2} \tag{2.49}$$

The analysis can be extended to describe several crucial properties of both typeI and typeII superconductors [20].

Recently, it was found [11] that $U(1)$ gauge symmetry is spontaneously broken below a crtical ratio of temperature and chemical potential near black hole horizons in asymptotically AdS spacetime. Consider the Ginzburg-Landau system (2.40) in $3+1$ dimensions and add gravity with negative cosmological constant to the system.

$$\mathcal{L} = R + \frac{6}{L^2} - \frac{1}{4}F_{ab}F^{ab} - g^{ab}(D_a\Psi)^\dagger D_b\Psi - V(|\psi|) \tag{2.50}$$

Background gauge field is set to $A_a = (A_t, \vec{0})$. Upon expanding the covariant derivatives, the Lagrangian becomes

$$\mathcal{L} = \text{kinetic terms} - V_{eff}(\psi) \tag{2.51}$$

with the effective potential $V_{eff}(\psi) = V(\psi) - |g_{tt}|A_t^2\psi^\dagger\psi$. The gauge potential thus gives a negative contribution to the mass matrix,and when it exceeds certain critical value, the perturbative vacuum will no longer be stable and the field ψ will condense. As soon as the field develops nonzero expectation value, the $U(1)$ symmtery is spontaneously broken. In our holographic description of superconducting phase transition we will use (2.50) as our gravity dual.

Chapter 3

Holography and AdS/CFT Correspondance

In the late 1960s it was found that a theory based on one dimensional extended object, rather than point particles, can account for various features of strong nuclear forces and strongly interacting particles. Here specific particle states were treated as specific modes of oscillation of this single object called strings. Unlike point particles, whose trajectories in space time are one dimensional curves called world lines, strings are one dimensional objects sweeping two dimensional areas through spacetime called world sheets. These objects admit two different topologies: a closed string has no endpoints and is topologically equivalent to a circle while an open string is equivalent to a line segment. Due to various technical problems and rise of Quantum Chromodynamics, the theory fell out of fervor. Nevertheless, research on dynamics of string like objects continued during 1970s, leading to formulation of (unphysical) bosonic string theory in 26 dimensions. It turned out that including fermions in the theory requires supersymmetry: a proposed symmetry of nature relating two basic classes of elementary particles bosons and fermions *. Interest was revived during 1980s when it was realised that string theory was capable of describing all elementary

*Supersymmetry is invented in two contexts at once: in ordinary particle field theory and as a consequence of introducing fermions into string theory particles as well as the interactions between them. The breakthrough came in 1984 with the discovery that quantum mechanical consistency of a ten dimensional theory with $\mathcal{N} = 1$ supersymmetry requires a local Yang-Mills gauge symmetry based on one of the two possible Lie algebras: $SO(32)$ and $E_8 \times E_8$. During this period, called first superstring revolution, five consistent string theories, all living on ten dimensions, were put forward. Excitations of a closed string can be decomposed into right and left moving plane waves. The supersymmetries associated with these modes can have either same or opposite handedness. In superstring formalism two possible handedness among modes of a string led to two different superstring theories: type IIA and type IIB. A third possibility let to type I superstring theory, which can be derived from type IIB. String theories were also constructed by hybridizing 26-dimensional bosonic string theory and 10 dimensional superstring theory as right and left moving modes. These are known as heterotic $SO(32)$ and heterotic $E_8 \times E_8$ string theories. In the early 1990s, strong evidences were found that the different superstring theories were different limits of a new 11-dimensional theory called M-theory sparking the second superstring revolution. Moreover, it was found that at low energies M-theory can be approximates by 11 dimensional supergravity: a supersymmetric classical gravity theory. Two powerful features of discoveries during this period are dualities and D-branes. It was found that different string theories reside on perturbative corners of a larger coupling space and related to each other via various dualities. For instance T-duality relates the two type II theories and two heterotic theories. In essence it establishes a correspondence between a string propagating on a large circle and a string propagating on a small circle. Study of T-duality on open string reveals that upon compactification the endpoints of an open string do not move in the direction along the dimension being compactified. The two endpoints freely move on fixed hyperplanes, orthogonal to compactified dimension separated by integral multiples of periodicity of dual dimension. These hypersurfaces, called Dirichlet branes or D-branes, revealed themselves as fundamental objects to our understanding of superstring theories and M-theory. One implication is that the quantum field theories of Yang-Mills type reside on the world volume of D-branes. The Yang-Mills fields arise from massless modes of open string attached to a D-brane. On the other hand D-branes were used to count the quantum states of an extreme black hole, matching the results of black hole thermodynamics proposed earlier by Bekenstein and Hawking. Another kind of duality, called S-duality, relates string coupling constant g_s to $1/g_s$. For instance it relates type I superstring theory to $SO(32)$ superstring theory and type II superstring theory to itself.

In 1998, Maldacena [8] has made a remarkable suggestion concerning the large N limit not of conventional $SU(N)$ gauge theories but of some of their conformal invariant cousins. According to this proposal,the large N limit of a conformal invariant theory in $d-1$ dimensions is governed by supergravity (and string theory) on d-dimensional AdS space (often called AdS_d) times a compact manifold which in the maximally supersymmetric cases is a sphere. There has also been a discussion of the flow to conformal field theory in some cases.

An important example to which this discussion applies is $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, with gauge group $SU(N)$ and coupling constant g_{YM}. This theory is equivalent to Type IIB superstring theory on $AdS_5 \times \mathbb{S}^5$, with string coupling constant g_{st} proportional to g^2_{YM}, N units of five-form flux on $\mathbb{S}^5$, and radius of curvature $(g^2_{YM}N)^{1/4}$. In the large N limit with $x = g^2_{YM}N$ fixed but large, the string theory is weakly coupled and supergravity is a good approximation to it. So the hope is that for large N and large x, the $\mathcal{N} = 4$ theory in four dimensions is governed by the tree approximation to supergravity.

The black holes under consideration [8] have near-horizon AdS geometries, and for our purposes it will suffice to work on the AdS spaces. AdS space has many unusual properties. It has a boundary at spatial infinity as a result of which quantization and analysis of stability are not straightforward. As we describe in this section, the boundary M_d of AdS_{d+1} is in fact a copy of d-dimensional Minkowski space (with some points at infinity added); the symmetry group $SO(2, d)$ of AdS_{d+1} acts on M_d as the conformal group. The fact that $SO(2, d)$ acts on AdS_{d+1} as a group of ordinary symmetries and on M_d as a group of conformal symmetries means that there are two ways to get a physical theory with $SO(2, d)$ symmetry: in a relativistic field theory (with or without gravity) on AdS_{d+1}, or in a conformal field theory on M_d. The possible relation of field theory on AdS_{d+1} to field theory on M_d has been a subject of long interest.The main idea in [8] was not that supergravity, or string theory,on AdS_{d+1} should be supplemented by singleton (or other) fields on the boundary, but that a suitable theory on AdS_{d+1} would be equivalent to a conformal field theory in d dimensions. [10]

This chapter is organized as follows. In the first section we will review some fundamental properties of AdS space and black solutions in AdS space. In the next section we will review the symmetry structure and quantization of conformal field theories. In the following section we will establish the bulk/boundary correspondence.

3.1 AdS Black Holes

Anti-de-Sitter space has generally been regarded as of little physical interest for two reasons. First it is the solution of Einstein's equation with negetive cosmological constant, if interpreted as vacume energy, corresponds to negetive energy density. Second anti-de-Sitter has closed timelike curves. The latter can be removed by passing to universal covering space by creating infinite replicas, one for each full period of timelike coordinate, but, this is not globally hyperbolic. The Cauchy data on a spacelike surface determines the evolution of the system, only in a region bounded by a null hypersurface called Cauchy horizon [29]. Thus, one has to specify not only the initial configuration but also boundary conditions, which describe radiation which comes in from infinity. Despite these difficulties, there are indications that anti-de-Sitter space may have physical importance. Extended theories of supergravity have AdS as their ground state (most symmetric state). Moreover, Witten's proof of positive mass theorem [30] has been extended to AdS space [31, 32]. Here I give a brief overview of AdS space and black hole solutions that are asumptotically AdS space.

Let us consider vacume Einstein's equations with cosmological constant Λ in d dimensions.

$$G_{ab} + \Lambda g_{ab} = 0 \tag{3.1}$$

where, $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$. We consider the simplest class of solutions of (3.1), which are characterized by a high degree of symmetries. Such spaces admit maximum number $d(d+1)/2$ of independent Killing vectors, which are generators of infinitesimal isometries (ie coordinate. transformation that leaves the metric form invariant), and thus called Maximally symmetric spaces. These spaces are locally characterised by geometric condition,

$$R_{abcd} = \frac{R}{d(d-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{3.2}$$

where R is Ricci scalar curvature. Using (3.1) and (3.2) we see that,

$$R = \frac{2d}{(d-2)}\Lambda \tag{3.3}$$

So they are locally constant curvature solutions. They are Einstein Spaces which are locally homogeneous and isotropic about every point. If $\Lambda = 0$ we have flat-Minkowski spacetime., for $\Lambda > 0$ positively curved de Sitter spacetime, and $\Lambda < 0$ negetively curved anti de Sitter spacetime.

In general for d dimension we have $d(d+1)/2$ independent components of Einstein's tensor G_{ab} and thus, $d(d+1)/2$ algebriacally independent Einstein's equations.

But these $d(d+1)/2$ components are related by d differential Bianchi identities ($G^a_{b;a} = 0$), and thus we have d degrees of freedom in $d(d+1)/2$ unknowns. These d degrees of freedom corresponding to arbitrary (but allowed) coordinate transform ($x \to x', g \to g'$), the metric in which also staisfies Einstein's equations. We can use this d degrees of freedom to choose d coordinate conditions that fixes this ambiguity. Maximally symmetric spaces admit Ricci tensor $R_{ab} \propto g_{ab}$ and Bianchi identities satisfy identically. Thus we have exactly $d(d+1)/2$ unknowns g_{ab} in $d(d+1)/2$ independent equations which uniquely determine the metric components.

A useful representation of d dimensional AdS spacetime is obtained by embedding d dimensional hyperboloid in $(d+1)$ dimensional flat spacetime.

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = L^2 \tag{3.4}$$

where $L > 0$ is the radias of curvature of above hyperboloid in $(d+1)$ dimensional flat spacetime, given by,

$$ds^2 = -dX_0^2 - dX_d^2 + \sum_{i=1}^{d-1} dX_i^2 \tag{3.5}$$

Clearly the translation invariance is broken by (3.4). Any element of Lorentz group $SO(2, d-1)$ will leave (3.4) and (3.5) invariant. Since $SO(2, d-1)$ has $d(d+1)/2$ Killing generators, it is the isometry group of AdS.

Another property useful to us is that AdS space has conformal boundary (ie it admits an equivalance class of metrics related via stretching and shrinking of coordinates, namely Conformal Transformations) at the hyperboloid infinity. If we rescale all coordinates $X_\mu \to \Lambda X_\mu$ then (3.4) becomes,

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \frac{L^2}{\Lambda^2} \tag{3.6}$$

The limit $\Lambda \to \infty$ defines the boundary as,

$$X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = 0 \tag{3.7}$$

If $X_0 \neq 0$ we can divid (3.7) through X_0 and rescaling $X_\mu \to \frac{X_\mu}{X_0}$ gives,

$$-X_d^2 + \sum_{i=1}^{d-1} X_i^2 = 1 \tag{3.8}$$

This is $(d-1)$ dimensional de Sitter space $(\mathbb{R} \times \mathbb{S}^{d-2})$ in d dimensional flat spacetime. If $X_0 = 0$ then (3.7) represents a sphere of $(d-2)$ dimension in $(d-1)$ dimensional flat spacetime.Together the boundary is maximally symmetric space $\mathbb{S}^1 \times \mathbb{S}^{d-2}$ and preserves $SO(1, d-1)$. In addition, it has d more transformation, 1 dilatation (generated by $D = -iX^\mu \partial_\mu$) and $(d-1)$ special conformal transformations (generated by $k_i = i(X^2\partial_i - 2X_iX^j\partial_j)$. So the symmetry group is Conformal Group $C(1, d-2)$ which is isomorphic to symmetry group $SO(2, d-1)$ of bulk AdS.

Now we briefly discuss the parameterisation of AdS space in d dimensions. The space given by (3.4) can be paremeterised by $X_0 = r_1 \cos(t)$, $X_d = r_1 \sin(t)$, $\sum_{i=1}^{d-1} X_i^2 = r_2^2$, so that (3.4) becomes $r_1^2 - r_2^2 = L^2$. Further putting $r_1 = L\cosh(u/L)$, $r_2 = L\sinh(u/L)$ in (3.5) with representing $(d-2)$ sphere in spherical coordinate $dK_{d-1}^2 = dr_2^2 + r_2^2 d\Omega_{d-2}^2$ we obtain line element of d-dimensional AdS space embedded in $(d+1)$ dimensional flat spacetime, parametrized by d coordinates $(t, u, \theta_1, ..., \theta_{d-2})$,

$$ds^2 = -(dr_1^2+r_1^2dt^2)+dr_2^2+r_2^2d\Omega_{d-2}^2 = -L^2\cosh^2(u/L)dt^2+du^2+L^2\sinh^2(u/L)d\Omega_{d-2}^2 \tag{3.9}$$

where $d\Omega_k^2 = d\theta_k^2 + \sin^2(\theta_k)d\Omega_{k-1}^2$.This is a global parameterisation of AdS with boundary at $u \to \infty$ and a solution to Einstein's equation (3.1) with cosmological constant,

$$\Lambda = -\frac{(d-1)(d-2)}{2L^2} \tag{3.10}$$

Note that the timelike coordinate $t \in [-\pi, +\pi]$ is an angular coordinate, which implies AdS is a spacetime with closed timelike curves. This is avoided by uncompactifying timelike coordinate by taking infinite copies of this hyperboloid, and map timelike coordinate of each hyperboloid into a noncompact coordinate $t \in [-\infty, +\infty]$.

We conclude this section with a discussion of black holes solutions in asymptotic AdS spacetimes. Consider, in d dimensional spacetime, a maximally symmetric $d-2$ dimensional spacelike section. They are classified (by normal curvature κ) as representing locally Euclidian ($\kappa = 0$), Spherical ($\kappa = 1$) and hyperbolic sections ($\kappa = -1$).

$$dK_{n-1}^2 = \frac{d\chi^2}{1-\kappa\chi_2} + \chi^2 d\Omega_{n-2}^2 \equiv L^2 h_{ij}dx^i dx^j \tag{3.11}$$

A spacetime with such maximally symmetric spacelike section, the general solution of Einstein's equation (3.1) admits locally timelike Killing vectors (Generalised Birkoff's theorem). The static Black hole solution reads [33],

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 h_{ij}dx^i dx^j \tag{3.12}$$

where,

$$V(r) = \kappa - \frac{\mu}{r^{d-3}} + \frac{r^2}{L^2} \tag{3.13}$$

The metric h_{ij} are functions of x^i only and referred to as horizon metric.We take the horizon to be compact manifold denoted by M^{d-2}.The AdS curvature L related to cosmological constant by (3.10). Here μ is an integration constant and can written as, $\mu = \omega_d M$, where,

$$\omega_d \equiv \frac{16\pi G}{(d-2)Vol(M^{d-2})} \tag{3.14}$$

Here $Vol(M^{d-2}) = \int d^{d-2}x\sqrt{h}$, so that M has a dimension of inverse length. This form of solution satisfies Einstein's Equation (3.1) with negetive cosmological constant,

$$R_{ab} = -\frac{d-1}{L^2} g_{ab} \tag{3.15}$$

provided the horizon is a constant curvature space of the form,

$$R_{ij}(h) = (d-3)\kappa h_{ij} \tag{3.16}$$

and thus can be of zero, positive and negetive curvature for $\kappa = 0, 1, -1$ respectively. We see that the $M = 0$ is a constant curvature spacetime and locally isometric to AdS. Though its topology depends on value of κ and thus topology of horizon. We note from (3.13) that the dominant behaviour of horizon is dominated by cosmological constant for any value of M. So we have a class of black hole solutions which are asymptotically locally anti de Sitter, for all values of M. The horizon r_h of these black holes are zeros of potential $V(r)$,

$$V(r_h) = 0 \tag{3.17}$$

To have a black hole interpretation we want the metric to describe the exterior of a black hole with a non degenerate horizon. This is achieved if the polynomial $V(r)r^{d-3}L^2$ has a simple positive root r_h, such that, $V(r) > 0$ for all $r > r_h$. For $\kappa = 0$, there is always a simple positive root of $V(r)r^{d-3}L^2$ given by $r_h = (\omega_d M L^2)^{1/(d-1)}$ with $V(r) > 0$ for all $r > r_h$.So we have black hole solutions with toroidal topology.

For $\kappa = 1, -1$ analysis of existance of simple positive roots are more involved and can be found here [33]. In the next subsection we will see that the for $\kappa = 1$ the black hole solution exists only above a critical temperature, while for $\kappa = 1, -1$ there is no such minimum temperature. We may use any one of these solutions which has an acceptable horizon located at r_h. The parameter M is specified in terms of r_h as,

$$M = \frac{r_h^{d-3}}{\omega_d}(\kappa + \frac{r_h^2}{L^2}) \tag{3.18}$$

Classical black holes behave like thermodynamical objects characterised by a temperature and entropy. From a thermodynamic point of view consider the partition function,

$$Z = Tr\left(e^{-\beta H}\right) \tag{3.19}$$

where H is the Hamiltonian of the system. Since the quantum mechanical evolution by a time interval t is given by e^{-iHt}, the trace corresponds imposing a periodicity β in Euclidian time. So the temperature T is periodic in Euclidian time with a period of β^{-1}. The temperature of a black hole is determined by analytic continuation to Euclidian time and examin the periodicity of this coordinate. Following this process for metric (3.12), the Hawking temperature of the AdS-Scwarzschild black hole reads,

$$T = \frac{(d-1)r_h^2 + (d-3)\kappa L^2}{4\pi L^2 r_h} \tag{3.20}$$

In gauge/gravity duality Hawking temperature of the black is identified with the temperature of dual field theory.

3.2 Conformal Field Theories

Symmetry principles, and in particular Lorentz and Poincare invariance, play a major role in our understanding of quantum field theory. It is natural to look for possible generalizations of Poincare invariance in the hope that they may play some role in physics; in [34] it was argued that for theories with a non-trivial S-matrix there are no such bosonic generalizations. An interesting generalization of Poincare invariance is the addition of a scale invariance symmetry linking physics at different scales (this is inconsistent with the existence of an S-matrix since it does not allow the standard definition of asymptotic states). Many interesting field theories, like Yang-Mills theory in four dimensions, are scale-invariant; generally this scale invariance does not extend to the quantum theory (whose definition requires a cutoff which explicitly breaks scale invariance) but in some special cases (such as the $d = 4, \mathcal{N} = 4$ supersymmetric Yang-Mills theory) it does, and even when it does not (like in QCD) it can still be a useful tool . It was realized in the past 30 years that field theories generally exhibit a renormalization group flow from some scale-invariant (often free) UV fixed point to some scale-invariant (sometimes trivial) IR fixed point, and statistical mechanics systems also often have non-trivial IR scale-invariant fixed points. Scale invariant field theories are important as possible end points of renormalization group flow in the space of cut-off effective field theories.

It is widely believed that unitary interacting scale-invariant theories are always invariant under the full conformal group, which is a simple group including scale invariance and Poincare invariance. This has only been proven in complete generality for two dimensional field theories [35, 37], but there are no known counter-examples. In this section we will review the symmetry structure of conformal group and introduce local conformal field operators $\mathcal{O}(x)$ as representations of that symmetry. We will further discuss an efficient way to obtain correlation functions in CFT by taking functional derivative of partition function with respect to some suitable classical source.

We consider the space $\mathbb{R}^d$ with flat metric $g_{ab} = \eta_{ab}$ of signature (p, q) with line element $ds^2 = g_{ab}dx^a dx^b$. By definition, the conformal group is the subgroup of coordinate transformation $x \to x'$, that leaves the metric invariant up to a scale change,

$$g_{ab}(x) \to g'_{ab}(x') = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}(x) = \Omega(x) g_{ab}(x) \tag{3.21}$$

Such transformations, preserve the angle between two vectors. If two vectors transform as $v \to v'$ and $w \to w'$ under a coordinate transformation $x \to x'$ then,

$$\frac{v' \cdot w'}{\sqrt{v'^2 w'^2}} = \frac{v \cdot w}{\sqrt{v^2 w^2}}$$

The infinitesimal generators of conformal group can be obtained by infinitesimal coordinate transformation $x^a \to x^a + \epsilon^a$, under which,

$$ds^2 \to ds^2 + (\partial_a \epsilon_b + \partial_b \epsilon_a) dx^a dx^b$$

To satisfy (3.21) the term $(\partial_a \epsilon_b + \partial_b \epsilon_a)$ must be proportional to η_{ab}. The proprtionality constant is detrmined by tracing both side by η_{ab}. Thus we get,

$$\partial_a \epsilon_b + \partial_b \epsilon_a = \frac{2}{d}(\partial \cdot \epsilon)\eta_{ab} \tag{3.22}$$

So the conformal factor is $\Omega(x) = 1 + \frac{2}{d}(\partial \cdot \epsilon)$. From (3.22) we obtain,

$$\{\eta_{ab}\Box + (d-2)\partial_a \partial_b\}\partial \cdot \epsilon = 0 \tag{3.23}$$

Both (3.22) and (3.23) requires the third derivatives of ϵ must vanish, so that ϵ is at most quadratic in x.

For zeroth order in x,

$$\epsilon^a = a^a \ \ : \text{Ordinary translation independent of } x$$

For first order in x,

$$\epsilon^a = \omega^a_b x^b \quad (\text{where}, \omega_{ab} = -\omega_{ba}) \ : \text{Rotation about } a\text{-th axis}$$

$$\epsilon^a = \lambda x^a \ \ : \text{Scale transformation}$$

For second order in x,

$$\epsilon^a = b^a x^2 - 2x^{as} b \cdot x \ \ : \text{Special conformal transformation}$$

Note that for special conformal transformation, $x \to x^a(1 - 2b \cdot x) + b^a x^2$, and $x^2 \to x^2(1 - 2b \cdot x)$ up to first order in ϵ, and thus, this transform up to first order is a composition of an inversion $x^a \to \frac{x^a}{x^2}$ and translation $\frac{x^a}{x^2} \to \frac{x^a}{x^2} + b^a$. The algebra generated by (no sum on a) $\equiv a^a\partial_a, \omega^a_b \epsilon^b \partial_a, \lambda x \cdot \partial$, and $b^a(x^2\partial_a - 2x^a x \cdot \partial)$, (a total of $(p+q) + \frac{1}{2}(p+q)(p+q-1) + 1 + (p+q) = \frac{1}{2}(p+q+1)(p+q+2)$ generators) is isomorphic to $SO(p+1, q+1)$.

Integrating to finite conformal transformation, $a^a\partial_a, \omega^a_b \epsilon^b \partial_a$ generate Pincare group,

$$x \to x + a, \ \ x \to \Lambda x \ \ (\text{where } \Lambda^a_b \in SO(p,q)) \ \text{ with } \Omega(x) = 1 \tag{3.24}$$

In addition we have dilatation,

$$x \to \lambda x \ \text{ with } \Omega(x) = \lambda^{-2} \tag{3.25}$$

and special conformal transformations,

$$x \to \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2} \quad \text{with } \Omega(x) = (1 + 2b \cdot x + b^2 x^2)^2 \tag{3.26}$$

The corresponding generators $P_a = -i\partial_a$, $M_{ab} = i(x_a\partial_b - x_b\partial_a)$, $D = -ix^a\partial_a$ and $K_a = -i(2x_a x^b \partial_b - x^2\partial_a)$ obey the following algebra,

$$[D, K_a] = iK_a\ ,\ [D, P_a] = -iP_a\ ,\ [P_a, K_b] = 2i(M_{ab} - g_{ab}D) \tag{3.27}$$

while other commutators either vanish or follow from rotational invariance. Note that the Poincare group together with dilation forms a subgroup of the full conformal group. This means that a theory invariant under translation,rotations and dilations are not necessarily invariant under special conformal transformation. This algebra is isomorphic to the algebra of $SO(d, 2)$, and can be put in the standard form of the $SO(d, 2)$ algebra (with signature $-, +, +, \cdots, +, -$) with generators J_{ab} ($a, b = 0, \cdots, d + 1$) by defining [†]

$$J_{ab} = M_{ab}; \qquad J_{ad} = \frac{1}{2}(K_a - P_a); \qquad J_{a(d+1)} = \frac{1}{2}(K_a + P_a); \qquad J_{(d+1)d} = D. \tag{3.28}$$

To analyze the constraints imposed by conformal invariance on N-point functions of a quantum theory, let us consider the Jacobian,

$$|\frac{\partial x'}{\partial x}| = \frac{1}{\sqrt{det g'_{ab}}} = \Omega^{-d/2} \tag{3.29}$$

For dilatation (3.25) and (3.26) this reads,

$$|\frac{\partial x'}{\partial x}| = \lambda^d \quad \text{and } |\frac{\partial x'}{\partial x}| = (1 + 2b \cdot x + b^2 x^2)^{-d} \tag{3.30}$$

[†] In the special case of $d = 2$ the conformal group is larger, and in fact it is infinite dimensional

Let there exist a set of fields $\{A_{\Delta_i}\}$ (where index Δ_i specifies different fields) such that members of a subset $\{\mathcal{O}_{\Delta_i}\} \subset \{A_{\Delta_i}\}$, under global conformal transformations $x \to x'$, transform as,

$$\mathcal{O}_{\Delta_i}(x) \to |\frac{\partial x'}{\partial x}|^{\Delta_i/d}\mathcal{O}_{\Delta_i}(x') \text{ where } \Delta_i \text{ is the scaling dimension of } \mathcal{O}_{\Delta_i} \tag{3.31}$$

such that, the correlation functions statisfy,

$$\begin{aligned}\langle\mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle \equiv \frac{\int\prod[D\mathcal{O}_\Delta]\mathcal{O}_{\Delta_1}\cdots\mathcal{O}_{\Delta_n}e^{-S[\mathcal{O}_\Delta]}}{\int\prod[D\mathcal{O}_\Delta]e^{-S[\mathcal{O}_\Delta]}} \\ = |\frac{\partial x'}{\partial x}|^{\Delta_1/d}\cdots|\frac{\partial x'}{\partial x}|^{\Delta_n/d}\langle\mathcal{O}_{\Delta_1}(x'_1)\cdots\mathcal{O}_{\Delta_n}(x'_n)\rangle\end{aligned} \tag{3.32}$$

where $S[\mathcal{O}_\Delta]$ is the action of the system. The members of this subset $\{\mathcal{O}_{\Delta_i}\}$ of fields are called "quasi-primary" fields. The rest of the members of $\{A_i\}$ can be expressed as linear combination of the quasi-primary fields and their derivatives. We also assume the existance of a vacume state $|0\rangle$ invariant under global conformal group. The covariance property (3.32) imposes several restrictions on N-point correlation functions.

One of the basic properties of conformal field theories is the one-to-one correspondence between local operators $\mathcal{O}$ and states $|\mathcal{O}\rangle$ in the radial quantization of the theory. In radial quantization, we foliate Euclidian $\mathbb{R}^d$ by $(d-1)$-spheres, $\mathbb{S}^{d-1}$, concentric at origin, and define Hilbert space of states of CFT at a given radial slice, where the time coordinate is chosen to be the radial direction in $\mathbb{R}^d$, with the origin corresponding to past infinity, so that the field theory lives on $\mathbb{R}\times\mathbb{S}^{d-1}$. The dilatation operator D then generates the evolution of states of CFT at a given radial distance. The Hamiltonian in this quantization is the operator $J_{0(d+1)}$ mentioned above. An operator $\mathcal{O}$ can then be mapped to the state $|\mathcal{O}\rangle = \lim_{x\to 0}\mathcal{O}(x)|0\rangle$. Equivalently, the state may be viewed as a functional of field values on some ball around the origin, and then the state corresponding to $\mathcal{O}$ is defined by a functional integral on a ball around the origin with the insertion of the operator $\mathcal{O}$ at the origin. The inverse mapping of states to operators proceeds by taking a state which is a functional of field values on some ball around the origin and using conformal invariance to shrink the ball to zero size, in which case the insertion of the state is necessarily equivalent to the insertion of some local operator.

We can classify local operators $\mathcal{O}_\Delta(x^a)$ by their tranformation properties of the little group $SO(d) \times SO(2) \subset SO(d,2)$ of conformal group. In radial quantization, (and in Euclidian signature), the $SO(d)$ irresducible representation is the spin of the field, while the charge under the $SO(2)$ subgroup is scaling dimension, Δ, of the field,

$$\mathcal{O}_\Delta(\lambda x^a) = \lambda^{-\Delta}\mathcal{O}_\Delta(x^a) \Leftrightarrow [D, \mathcal{O}_\Delta(0)] = -i\Delta\mathcal{O}_\Delta(0) \tag{3.33}$$

The commutation relations (3.27) imply that the operator P_a raises the dimension of the field, while the operator K_a lowers it. In unitary field theories there is a lower bound on the dimension of fields (for scalar fields it is $\Delta \geq (d-2)/2$ which is the dimension of a free scalar field), and, therefore, each representation of the conformal group which appears must have some operator of lowest dimension, which must then be annihilated by K_a (at $x = 0$). Such operators are called Primary operators. By translating such operators to arbitrary position, it follows that they obey the following commutation relations,

$$[P_a, \mathcal{O}_\Delta(x)] = i\partial_a\mathcal{O}_\Delta(x)$$

$$[M_{ab}, \mathcal{O}_\Delta(x)] = i\{(x_a\partial_b - x_b\partial_a) + \Sigma^R_{ab}\}\mathcal{O}_\Delta(x)$$

$$[D, \mathcal{O}_\Delta(x)] = i(x^a\partial_a - \Delta)\mathcal{O}_\Delta(x)$$

$$[K_a, \mathcal{O}_\Delta(x)] = i\{(x^2\partial_a - 2x_a x^b\partial_b + 2x_a\Delta) - 2x^b\Sigma^R_{ab}\}\mathcal{O}_\Delta(x)$$

where Σ^R_{ab} are representation of irrudicible spin R of the primary which acts on its spin indices.

A general property of local field theories is the existence of an *operator product expansion* (OPE). As we bring two operators $\mathcal{O}_{\Delta_1}(x)$ and $\mathcal{O}_{\Delta_2}(y)$ to the same point, their product creates a general local disturbance at that point, which may be expressed as a sum of local operators acting at that point; in general all operators with the same global quantum numbers as $\mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2}$ may appear. The general expression for the OPE is $\mathcal{O}_{\Delta_1}(x)\mathcal{O}_{\Delta_2}(y) \to \sum_n C_{12}^n(x-y)\mathcal{O}_n(y)$, where this expression should be understood as appearing inside correlation functions, and the coefficient functions C_{12}^n do not depend on the other operators in the correlation function (the expression is useful when the distance to all other operators is much larger than $|x-y|$). In a conformal theory, the functional form of the OPE coefficients is determined by conformal invariance to be $C_{12}^n(x-y) = c_{12}^n/|x-y|^{\Delta_1+\Delta_2-\Delta_n}$, where the constants c_{12}^n are related to the 3-point functions described above. The leading terms in the OPE of the energy-momentum tensor with primary fields are determined by the conformal algebra. For instance, for a scalar primary field $\mathcal{O}$ of dimension Δ in four dimensions,

$$T_{ab}(x)\mathcal{O}_\Delta(0) \propto \Delta\mathcal{O}_\Delta(0)\partial_a\partial_b(\frac{1}{x^2}) + \cdots \tag{3.34}$$

In principle all n-point correlators are determined by OPE's, since any n-point functions can be replaced by an infinite sum of $(n-1)$-point functions by using OPE for any two adjacent insertions. Thus, the CFT is completely described by the data $\{\Delta_i, spins, c_{ijk}\}$ for all the primaries (labelled i, j, k). However this data is highly constrained, and thus, an arbitrary set of data will not general define a consistent CFT. AdS/CFT provides a different approach to overcome these difficulties.

Consider the partition function,

$$Z[\tilde{\psi}_{\Delta_i}] = \langle exp(\int d^d x \tilde{\psi}_{\Delta_i}(x) \mathcal{O}_{\Delta_i}) \rangle_{CFT} \tag{3.35}$$

This is a functional of classical source $\tilde{\psi}_{\Delta_i}(x)$ associated with each field operator $\mathcal{O}_{\Delta_i}$ of CFT, which generates correlation functions by taking derivative of Z with respect to the sources,

$$\langle \mathcal{O}_{\Delta_1}(x_1) \mathcal{O}_{\Delta_2}(x_2) \cdots \rangle = \frac{\partial^n Z[\tilde{\psi}_{\Delta_i}]}{\partial \tilde{\psi}_{\Delta_1}(x_1) \partial \tilde{\psi}_{\Delta_2}(x_2) \cdots} |_{\tilde{\psi}_{\Delta_i}=0} \tag{3.36}$$

The conformal invariance of correlators is reflected in the conformal invariance of $Z[\tilde{\psi}_\Delta]$. In particular under scaling,

$$\int d^d x \tilde{\psi}_\Delta(x) \mathcal{O}_\Delta(x) = \int d^d(\lambda x) \tilde{\psi}_\Delta(\lambda x) \mathcal{O}_\Delta(\lambda x) = \lambda^{d-\Delta} \int d^d x \tilde{\psi}_\Delta(\lambda x) \mathcal{O}_\Delta(x) \tag{3.37}$$

So Z is invariant under above scaling tranformation, if, the source transforms as

$$\tilde{\psi}_\Delta(x) \rightarrow \lambda^{d-\Delta} \tilde{\psi}_\Delta(\lambda x) \tag{3.38}$$

In the next section we will discuss an elegant method of obtaining partition function $Z[\tilde{\psi}_\Delta]$ from a dual theory living on one higher space dimension.

3.3 AdS/CFT Correspondance

In the last couple of sections we have established that the d dimensional conformal boundary ∂AdS_d of a $d+1$ dimensional AdS space has the same isometry group as CFT on a manifold of d dimensions. So ∂AdS is suitable for a space to define a d dimensional CFT. Introducing a new parametrisation $x = (z, t, \vec{x})$ with $z \rightarrow e^{-u/L}$

and uncompactifying the rescaling time coordinate in equation (3.9) we can bring the AdS_{d+1} in a simple form whose boundary is at $z = 0$.

$$ds^2 = \frac{L^2}{z^2}\left(-dt^2 + dz^2 + d\vec{x}^{\,2}\right) \qquad \vec{x} = (x^1, \cdots x^{d-1}) \tag{3.39}$$

Thus, AdS_{d+1} is conformal to upper half space $z > 0$ of $\mathbb{R}^{d+1}$ while its boundary $\partial\mathrm{AdS}_d$ is conformal to $\mathbb{R}^d$. Any generally covariant combinations tensor fields $\psi(z, t, \vec{x})$ on AdS will be conformal invariant. Any restriction of $\psi(z, t, \vec{x})$ on boundary $\partial\mathrm{AdS}_d = (z = 0, t, \vec{x})$ transforms as representation of conformal group and therefore a covariant combination of them will be also conformal invariant. In other words, existence of a boundary with the same symmtery group as its bulk superspace allows us to construct a partition function of a d dimensional CFT by taking covariant functions of boundary values of fields that reside in higher dimensional bulk space. Defining $\tilde{\psi}(t, \vec{x}) \equiv lim_{z\to 0}\psi(z, t, \vec{x})$ we write,

$$Z\left[\,\tilde{\psi}_\Delta(t, \vec{x})\,\right] = \int_{\psi|_\partial = \tilde{\psi}_\Delta} \mathcal{D}\psi(x)\, e^{-S[\psi(x)]} \tag{3.40}$$

It follows that if ψ behaves at boundary as,

$$\psi(z, t, \vec{x}) = z^{d-\Delta}\,\tilde{\psi}_\Delta(t, \vec{x}) + \mathbb{O}\left(z^{d-\Delta-1}\right) \tag{3.41}$$

then $\tilde{\psi}_\Delta$ transforms according to equation 3.38 and hence identified as a source of an operator of dimension Δ in boundary theory.

Chapter 4

Holographic Superconductors

Motivated by cuprate superconductors, a flurry of theoretical work have been done on quantum phases and quantum phase transitions in $2 + 1$ dimensional strongly correlated electron systems. A quantum phase transition is phase transition between different states of matter at $T = 0$. They are driven by quantum fluctuations associated with Heisenberg uncertainty principle rather than thermal fluctuations. Let g be the physical parameter driving the quantum phase transition. The quantum critical point g_c is a point in parameter space at which the phase transition takes place. At $T = 0$ but away from critical point, the system has a characteristic energy scale Δ and coherence length ξ associated with length scale over which correlations in the system are lost. We expect Δ to vanish and ξ to diverge at quantum critical point as,

$$\Delta \sim (g - g_c)^{\nu z} \quad ; \quad \xi \sim (g - g_c)^{-\nu} \tag{4.1}$$

The quantity z relating the behavior $\Delta \sim \xi^{-z}$ is called the dynamical scaling exponent. At the quantum critical point the system becomes invariant under rescaling of time and distance,

$$t \to \lambda^z t \quad ; \quad \vec{x} \to \lambda \vec{x} \tag{4.2}$$

Different z occur in different condensed matter system. We consider systems with $z = 1$ since in this case the system preserves Lorentz symmetry in d spacetime dimensions. In fact, together with scaling symmetry, they become part of the larger conformal symmetry group $SO(d, 2)$. Thus, quantum critical systems with scaling exponent one are suitable CFTs for applications of AdS/CFT correspondence. One class of problems are associated with strongly interacting quantum systems at finite temperature near quantum critical points. To this end an effective scale invariant field theory at critical point can be extended to nonzero T_c. Holographic superconductors are thus a class of materials characterised by strong interaction, proximity to a quantum phase transition. In gravity dual the supersymmetry as well as the conformal symmetry is broken by finite temperature and chemical potential.

The existence of holographic superconductors was established in [11, 15, 41]. From the (d dimensional) field theory point of view, superconductivity is characterised by the condensation of a, generically composite, charged operator $\mathcal{O}$ for low temperatures $T < T_c$. In the dual ($d + 1$ dimensional) gravitational description of the system, the transition to superconductivity is observed as a classical instability of a black hole in anti-de Sitter (AdS) space against perturbations by a charged scalar field ψ. The instability appears when the black hole has Hawking temperature $T = T_c$. For lower temperatures the gravitational dual is a black hole with a nonvanishing profile for the scalar field ψ. The AdS/CFT correspondence relates the highly quantum dynamics of the 'boundary' operator $\mathcal{O}$ to simple classical dynamics of the 'bulk' scalar field ψ [9, 10].

4.1 The model

In the superconductor we need a notion of temperature. On the gravity side, that role is played by a black hole. In gauge/gravity duality, the Hawking temperature of the black hole is identified with the temperature of the dual field theory. Since gauge/gravity duality traditionally requires that spacetime asymptotically approach anti de Sitter (AdS) space at infinity, we will be studying black holes in AdS. Unlike asymptotically flat black holes, these black holes have the property that at large radius, their temperature increases with their mass, i.e., they have positive specific heat, just like familiar nongravitational systems. There are also planar AdS black holes, which will be of most interest. These black holes always have positive specific heat.

In the superconductor, we also need a condensate. In the bulk, this is described by some field coupled to gravity. A nonzero condensate corresponds to a static nonzero field outside a black hole. This is usually called black hole "hair". So to describe a superconductor, we need to find a black hole that has hair at low temperatures, but no hair at high temperatures. More precisely, we need the usual Schwarzschild or Reissner-Nordstrom AdS black hole (which exists for all temperatures) to be unstable to forming hair at low temperature.

A simple solution to this problem was found by Gubser [11]. He argued that a charged scalar field around a charged black hole in AdS would have the desired property. Consider

$$S = \int d^4x\sqrt{-g}\left(R + \frac{6}{L^2} - \frac{1}{4}F_{ab}F^{ab} - g^{ab}(D_a\Psi)^* D_b\Psi - m^2|\Psi^2|\right). \qquad (4.3)$$

where $D_a \equiv \partial_a - iqA_a$. This is just general relativity with a negative cosmological constant $\Lambda = -3/L^2$, coupled to a Maxwell field and charged scalar with mass m and charge q. For an electrically charged black hole, the effective mass of Ψ is $m^2_{eff} = m^2 + q^2 g^{tt} A_t^2$. But the last term is negative, so there is a chance that m^2_{eff} becomes sufficiently negative near the horizon to destabilize the scalar field. Furthermore, as one lowers the temperature of a charged black hole, it becomes closer to extremal which means that g_{tt} is closer to developing a double zero at the horizon. This means that $|g^{tt}|$ becomes larger and the potential instability becomes stronger at low temperature.

4.2 Condensate

Line element of a planar Schwarzschild anti-de Sitter black hole reads,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2)\,, \tag{4.4}$$

where

$$f = \frac{r^2}{L^2} - \frac{M}{r}\,. \tag{4.5}$$

L is the AdS radius and M determines the Hawking temperature of the black hole:

$$T = \frac{3M^{1/3}}{4\pi L^{4/3}}\,. \tag{4.6}$$

This black hole is 3+1 dimensional, and so will be dual to a 2+1 dimensional theory. In this background, we now consider a Maxwell field and a charged complex scalar field, with Lagrangian density*

$$\mathcal{L} = -\frac{1}{4}F^{ab}F_{ab} - V(|\Psi|) - g^{\mu\nu}(D_\mu\Psi)^* D_\nu\Psi\,. \tag{4.7}$$

For simplicity and concreteness, we will focus on the case

$$V(|\Psi|) = -\frac{2|\Psi|^2}{L^2}\,. \tag{4.8}$$

Although the mass squared is negative, it is above the Breitenlohner-Freedman bound [36] and hence does not induce an instability. It corresponds to a conformal coupled scalar in our background (4.4). This is a new source of instability. An extremal

*Introducing a gauge coupling $1/q^2$ in front of the $|F|^2$ term in the action is equivalent to rescaling the fields $\Psi \to q\Psi$ and $A_\mu \to qA_\mu$. Setting $q = 1$ is a choice of units of charge in the dual 2+1 theory.

Reissner-Nordstrom AdS black hole has a near horizon geometry $AdS_2 \times R^2$. The BF bound for AdS_{d+1} is $m_{BF}^2 = -d^2/4$. So scalars which are slightly above the BF bound for AdS_4, can be below the bound for AdS_2. This instability to forming neutral scalar hair is not associated with superconductivity (or superfluidity) since it doesn't break a $U(1)$ symmetry. At most it breaks a Z_2 symmetry corresponding to $\psi \to -\psi$. Its interpretation in the dual field theory is not clear.

We will work in a limit in which the Maxwell field and scalar field do not backreact on the metric. This limit is consistent as long as the fields are small in Planck units. Alternatively, this decoupled Abelian-Higgs sector can be obtained from the full Einstein-Maxwell-scalar theory considered in [11] through a scaling limit in which the product of the charge of the black hole and the charge of the scalar field is held fixed while the latter is taken to infinity. Thus we will obtain solutions of nonbackreacting scalar hair on the black hole.

Taking a plane symmetric ansatz, $\Psi = \Psi(r)$, the scalar field equation of motion is

$$\Psi'' + \left(\frac{f'}{f} + \frac{2}{r}\right)\Psi' + \frac{\xi^2}{f^2}\Psi + \frac{2}{L^2 f}\Psi = 0\,, \tag{4.9}$$

where the scalar potential $A_t = \xi$. With $A_r = A_x = A_y = 0$, the Maxwell equations imply that the phase of Ψ must be constant. Without loss of generality we therefore take Ψ to be real. The equation for the scalar potential ξ is the time component of the equation of motion for a massive vector field

$$\xi'' + \frac{2}{r}\xi' - \frac{2\Psi^2}{f}\xi = 0\,, \tag{4.10}$$

where $2\Psi^2$ is the, in our case, r dependent mass. The charged condensate has triggered a Higgs mechanism in the bulk theory. At the horizon, $r = r_0$, for $\xi\, dt$ to have finite norm, $\xi = 0$, and (4.9) then implies $\Psi = -3r_0\Psi'/2$. Thus, there is a two parameter family of solutions which are regular at the horizon. Integrating out to infinity, these solutions behave as

$$\Psi = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + \cdots . \tag{4.11}$$

and

$$\xi = \mu - \frac{\rho}{r} + \cdots . \tag{4.12}$$

For Ψ, both of these falloffs are normalizable [18], so one can impose the boundary condition that either one vanishes.† After imposing the condition that either $\Psi^{(1)}$ or $\Psi^{(2)}$ vanish we have a one parameter family of solutions. It follows from (4.10) that the solution for ξ is always monotonic: It starts at zero and cannot have a positive maximum or a negative minimum. Note that even though the field equations are nonlinear, the overall signs of ξ and Ψ are not fixed. We will take ξ to be positive and hence have a system with positive charge density. The sign of Ψ is part of the freedom to choose the overall phase of Ψ. Properties of the dual field theory can be read off from the asymptotic behavior of the solution. For example, the asymptotic behavior (4.12) of ξ yields the chemical potential μ and charge density ρ of the field theory. The condensate of the scalar operator $\mathcal{O}$ in the field theory dual to the field Ψ is given by

$$\langle \mathcal{O}_i \rangle = \sqrt{2}\Psi^{(i)} , \quad i = 1, 2 \tag{4.13}$$

with the boundary condition $\epsilon_{ij}\Psi^{(j)} = 0$. The $\sqrt{2}$ normalization simplifies subsequent formulae, and corresponds to taking the bulk-boundary coupling $\frac{1}{2}\int d^3x(\bar{\mathcal{O}}\Psi + \mathcal{O}\bar{\Psi})$. Note that $\mathcal{O}_i$ is an operator with dimension i. From this point on we will work in units in which the AdS radius is $L = 1$. Recall that T has mass dimension one, and ρ has mass dimension two so $\langle \mathcal{O}_i \rangle / T^i$ and ρ/T^2 are dimensionless quantities.

An exact solution to eqs (4.9,4.10) is clearly $\Psi = 0$ and $\xi = \mu - \rho/r$. It appears difficult to find other analytic solutions to these nonlinear equations. However, it is

†One might also imagine imposing boundary conditions in which both $\Psi^{(1)}$ and $\Psi^{(2)}$ are nonzero. However, if these boundary conditions respect the AdS symmetries, then the result is a theory in which the asymptotic AdS region is unstable [38].

straightforward to solve them numerically. We find that solutions exist with all values of the condensate $\langle\mathcal{O}\rangle$. However, as shown in figure 4.1, in order for the operator to condense, a minimal ratio of charge density over temperature squared is required. The right hand curve in figure 4.1 is qualitatively similar to that obtained in BCS

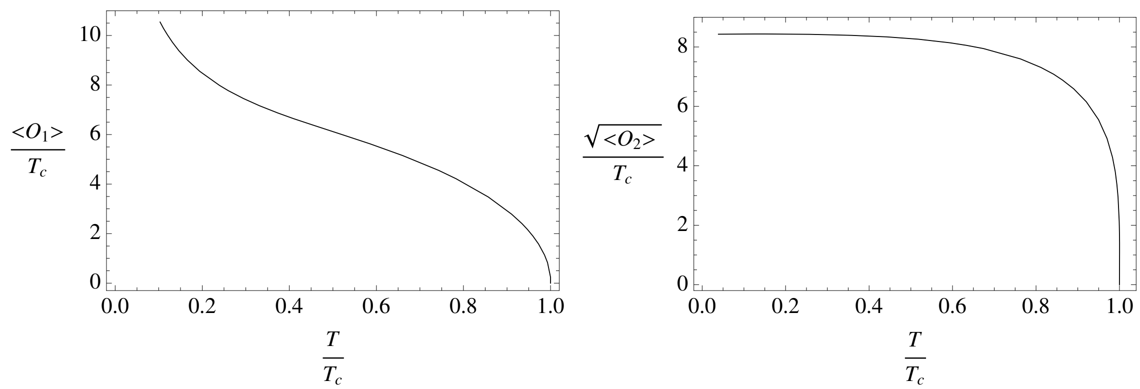

Figure 4.1: The condensate as a function of temperature for the two operators $\mathcal{O}_1$ and $\mathcal{O}_2$. The condensate goes to zero at $T = T_c \propto \rho^{1/2}$.

theory, and observed in many materials, where the condensate goes to a constant at zero temperature. The left hand curve starts similarly, but at low temperature the condensate appears to diverge as $T^{-1/6}$. However, when the condensate becomes very large, the backreaction on the bulk metric can no longer be neglected. At extremely low temperatures, we will eventually be outside the region of validity of our approximation.

By fitting these curves, we see that for small condensate there is a square root behaviour that is typical of second order phase transitions. Specifically, for one boundary condition we find

$$\langle\mathcal{O}_1\rangle \approx 9.3\, T_c\, (1 - T/T_c)^{1/2}\,, \quad \text{as} \quad T \to T_c\,, \tag{4.14}$$

where the critical temperature is $T_c \approx 0.226\rho^{1/2}$. For the other boundary condition

$$\langle \mathcal{O}_2 \rangle \approx 144\, T_c^2\, (1 - T/T_c)^{1/2}\,, \quad \text{as} \quad T \to T_c\,, \tag{4.15}$$

where now $T_c \approx 0.118\rho^{1/2}$. The continuity of the transition can be checked by computing the free energy. Finite temperature continuous symmetry breaking phase transitions are only possible in 2+1 dimensions in the large N limit (i.e. the classical gravity limit of our model), where fluctuations are suppressed. These transitions will become crossovers at finite N. Thus for $T < T_c$ a charged scalar operator, $\langle \mathcal{O}_1 \rangle$ or $\langle \mathcal{O}_2 \rangle$, has condensed. It is natural to expect that this condensate will lead to superconductivity of the current associated with this charge.

Chapter 5

Holographic Striped Superconductors

Superconductivity in copper oxides arise from doping of half filled Mott insulators. For example, consider the parent compound La_2CuO_4, which is an antiferromagnetic insulator at low temperatures. To turn La_2CuO_4 into a superconductor, the compound can be doped with strontium which has an effect of removing one electron from every lanthanium atom replaced with strontium. Once the doping concentration x in $La_{2-x}Sr_xCuO_4$becomes sufficiently large, the compound superconducts at low temperature. There exist an optimal doping x_0 at which the highest T_c is obtained. When $x > x_0$ the compound is said to be over doped and for $x < x_0$ it is called under doped. Many low temperature properties are compatible with BCS-like pairing of d-wave symmetry. In strong over doped regime the normal state behaves like Fermi liquid whose only instability is to superconductivity. In this regime the effective degrees of freedom become weakly coupled and the physics is relatively well understood. In contrast, the effective degrees of freedom are believed to be strongly interacting in under doped regime. In this regime superconducting to normal state phase transition involves disordering the phase of condensate. The electrons can still form pairs in normal state of compound. This region of temperature-doping phase space is called pseudogap region. The pseudogap regime in under doped compounds is still not understood. One fundamental problem is highly unconventional normal state that appears to violate Fermi liquid properties. The doping evolution of normal state Fermi surface and magnetic moments are still elusive. Moreover, Linear temperature dependence of resistivity around optimal doping above T_c and asymmetry between electron and hole doped materials pose further puzzles.

Both theoretical and experimental studies indicate presence of various competing orders in cuprates. Although, the most prominent ones are commensurate antiferromagnetic and superconducting order, incommensurate unidirectional spin density waves (SDW) and charge density waves (CDW), called stripes play an important role in these compounds. These orders are associated with modulation of spin and charge density respectively that break the discrete translational and rotational symmetry underlying the Cu_2O_4 plane. These states are first predicted in mean-field studies of Hubbard model and later observed in $La_{2-x}Ba_xCuO_4$and $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$. Since then signatures of stripes have been observed in many other high T_c materials.

The charge density wave (CDW) state is a state where the charge density oscillates around its average value as,

$$\rho(t,\vec{r}) = \rho_0(t,\vec{r}) + \rho_1(t,\vec{r})\, cos(\vec{Q}_c \cdot \vec{r}) \tag{5.1}$$

and spin density wave is a state with spatially modulated spin vector.

$$S^i(t,\vec{r}) = S^i_1(t,\vec{r})\, cos(\vec{Q}_s \cdot \vec{r}) \tag{5.2}$$

where $\vec{Q}_c$ and $\vec{Q}_s$ are charge and spin ordering vector respectively. In general the transition into a state with broken translational symmetry could either display a modulation in charge density only or in both charge and spin density. In low dimensional metals CDW forms as a consequence of electron-phonon interactions.

The nature of interplay between these unidirectional stripe orders and superconductivity is still poorly understood. In particular we may ask: *do stripes enhance or inhibit superconductivity?* . The critical temperature T_c as a function of doping follows a parabolic dome shape curve for multilayer compounds like $Bi_2Sr_2CaCu_2O_{8+\delta}$. In single layer compounds like $La_{2-x}Sr_xCuO_4$and $La_{2-x}Ba_xCuO_4$$T_c$ behaves similarly except a pronounced T_c suppression at $x = 1/8$ doping. This sharp dip on dome curve is called the "1/8 anomaly" (5.1). Since the stripe order is particularly strong near $x = 1/8$ doping in these compounds, the 1/8 anomaly is commonly attributed to stripes *competing* with superconductivity. Although, data shows the physics in that region is more complicated than competition of two orders. On the other hand few observations such as in spin density waves in $La_{2-x}Sr_xCuO_4$indicate that stripes *enhances* superconductivity. See [17] for a review.

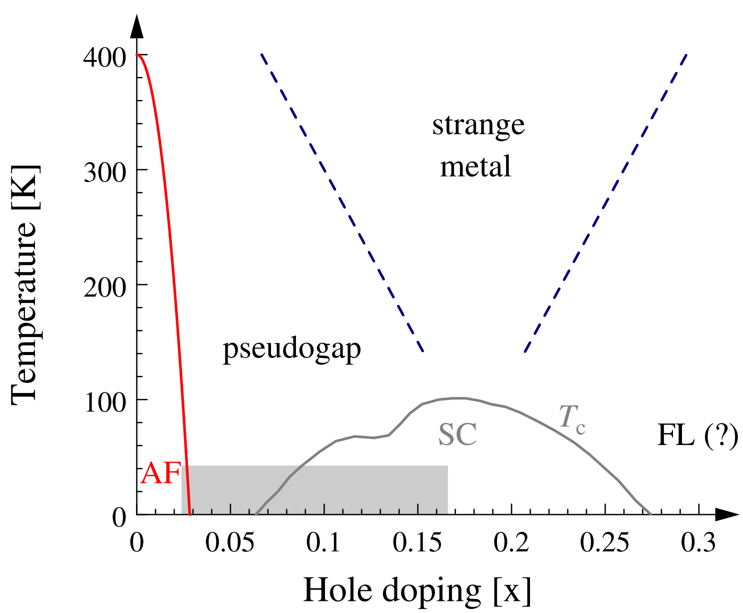

Figure 5.1: Temperature-Doping Phase diagram of cuprates.

In this thesis we study the interplay between CDW and superconductivity with a holographic model. We study the effect of CDW on T_c of a $2+1$ dimensional strongly coupled conformal system with planar topology [43, 44]. The effect of charge density is introduced phenomenologically by an externally modulated chemical potential with wave vector Q as in [40]. The T_c vs Q phase structure is studied for condensates of various scaling dimensions. It is found [44] that for a fixed chemical potential μ there is an *enhancement* of T_c upon turning on modulation. On the other end for large modulation either T_c has a power law behavior or there exists a critical modulation beyond which no phase transition takes place. Moreover, there exists a regime in parameter space of this theory in which T_c vs Q phase structure reproduces [58] the T_c vs doping phase structure of cuprates, namely the superconducting dome, provided there exists a monotonic relation between doping and CDW modulation.

5.1 The System

We are interested in studying a strongly coupled striped superconductor using the gauge/gravity duality. To this end, consider a $U(1)$ gauge potential A^a and a scalar field ψ charged under this potential living in a $3+1$-dimensional spacetime with negative cosmological constant $\Lambda = -3/L^2$. The scalar field is dual to the scalar order parameter of the superconductor, i.e., the condensate, while the $U(1)$ gauge field is dual to the four-current in the strongly coupled system. For simplicity, we shall adopt units in which $L = 1$, $16\pi G = 1$. To study the strong coupling regime of the superconductor, we only need to study the gravity theory at the classical level. In particular, we are interested in finding solutions to the classical equations of motion whose boundary values are related to the parameters of the superconductor. The action for this system is

$$S = \int d^4x\sqrt{-g}\left[R + 6 - \frac{1}{4}F^2 - |D_a\psi|^2 - m^2|\psi|^2\right], \tag{5.3}$$

where, $D_a = \partial_a - iqA_a$, $F_{ab} = \partial_a A_b - \partial_b A_a$ and $a, b \in \{t, r, x, y\}$. Here, m and q are the mass and charge of the scalar field, respectively.

The field equations consist of the Einstein equations,

$$R_{ab} - \frac{1}{2}g_{ab}R - 3g_{ab} = \frac{1}{2}T_{ab}\,, \tag{5.4}$$

where the stress-energy tensor is

$$\begin{aligned} T_{ab} &= F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd} \\ &+ D_a\psi(D_b\psi)^* + (a \leftrightarrow b) - g_{ab}\left[|D_a\psi|^2 + m^2|\psi|^2\right], \end{aligned} \tag{5.5}$$

the Maxwell equations,

$$\frac{1}{\sqrt{-g}}\partial_b(\sqrt{-g}F^{ab}) = J^a\,, \tag{5.6}$$

where the $U(1)$ current is

$$J^a = -i[\psi^* D^a\psi - \text{c.c.}]\,, \tag{5.7}$$

and the Klein-Gordon equation for the scalar field,

$$-\frac{1}{\sqrt{-g}}D_a(\sqrt{-g}g^{ab}D_b\psi) + m^2\psi = 0\,. \tag{5.8}$$

We are interested in finding a static black hole solution of flat conformal boundary which is sourced by a modulated chemical potential $\mu(\vec{x})$, where μ is a given spatially dependent function. The presence of this modulated chemical potential gives rise to inhomogeneities in the system. However, it should be emphasized that this is a phenomenological description of inhomogeneities. The chemical potential is an effective potential resulting from interactions within the system. Ultimately, one would like to understand the dynamical emergence of the modulated potential and attendant inhomogeneities. However, here, we are only interested in the consequences of the presence of the modulated potential, which we treat as fixed.

Let (x, y) be the Cartesian spatial coordinates of the two-dimensional conformal boundary. We concentrate on the case in which μ only depends on one of the coordinates, which is chosen to be x. For definiteness, we only consider the case in which only a homogeneous term and a single oscillating mode are present,

$$\mu(x) = \mu(1 - \delta) + \mu\delta \cos Qx \tag{5.9}$$

To find a solution to the coupled Einstein-Maxwell equations 5.4 and 5.6, consider the metric *ansatz*

$$ds^2 = -r^2 e^{-\alpha} dt^2 + e^{\alpha} \frac{dr^2}{r^2} + r^2 e^{-\beta} \left[e^{-\gamma} dx^2 + e^{\gamma} dy^2 \right] , \tag{5.10}$$

where α, β, and γ are functions of (r, x). The boundary is at $r \to \infty$ and the horizon is at $r = r_+$, where r_+ is an arbitrary parameter. For a flat conformal boundary, we require α, β, $\gamma \to 0$, as $r \to \infty$, and in fact, we find $\alpha \sim \mathcal{O}(r^{-3})$ while β and $\gamma \sim \mathcal{O}(r^{-4})$.

For the $U(1)$ potential, we fix the gauge so that $A_r = A_x = A_y = 0$ and $A_t = A_t(r, x)$ with $A_t = 0$ at the horizon (for a finite norm, $A_a A^a < \infty$), whereas at the boundary,

$$A_t(r, x)\Big|_{r\to\infty} = \mu(x) \tag{5.11}$$

We shall solve the Einstein-Maxwell equations 5.4 and 5.6 perturbatively by expanding around the Schwarzschild solution, which is obtained as $\mu \to 0$. This corresponds to the probe limit in which the scalar charge $q \to \infty$ so that the product $q\,\mu$ remains finite. Near the critical temperature, we have a radius of the horizon of the same order as $q\mu$ ($r_+ \sim q\mu$), so an expansion in μ is equivalent to an expansion in $1/q$. More precisely, the expansion is in the dimensionless parameter

$$\left(\frac{\mu}{r_+}\right)^2 \sim \frac{1}{q^2}\ , \tag{5.12}$$

which is the only parameter in the Einstein-Maxwell system (since the vector potential enters quadratically). This expansion is valid for large black holes (or small chemical potential), or more precisely for

$$\mu \lesssim r_+\,. \tag{5.13}$$

Expanding in the small dimensionless parameter (5.12), we have

$$\begin{aligned} A_t &= A_t^{(0)} + \left(\frac{\mu}{r_+}\right)^2 A_t^{(1)} + \dots \\ \alpha &= \alpha^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \alpha^{(1)} + \dots \\ \beta &= \beta^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \beta^{(1)} + \dots \\ \gamma &= \gamma^{(0)} + \left(\frac{\mu}{r_+}\right)^2 \gamma^{(1)} + \dots \end{aligned} \tag{5.14}$$

and consequently, the expansions of the metric, Ricci tensor and $U(1)$ field strength and stress-energy tensor, respectively,

$$\begin{aligned} g_{ab} &= g_{ab}^{(0)} + \left(\frac{\mu}{r_+}\right)^2 g_{ab}^{(1)} + \dots \\ R_{ab} &= R_{ab}^{(0)} + \left(\frac{\mu}{r_+}\right)^2 R_{ab}^{(1)} + \dots \\ F_{ab} &= \left(\frac{\mu}{r_+}\right)^2 F_{ab}^{(0)} + \dots \\ T_{ab} &= \left(\frac{\mu}{r_+}\right)^2 \mathcal{T}_{ab}^{(0)} + \dots \end{aligned} \tag{5.15}$$

5.2 Mean-Field Analysis

In the mean-field treatment we will neglect the backreaction of the scalar field and the gauge field onto the metric. This is valid when scale invariant quantities, such as $G_N\mu^2$, where μ is the chemical potential, are small. As usual, this can be achieved by sending G_N to zero while keeping everything else fixed.

At zeroth order, the Einstein-Maxwell equations read

$$R^{(0)\,a}{}_b + 3\delta^a{}_b = 0\,, \qquad \partial_b\left(\sqrt{-g^{(0)}}F^{(0)ab}\right) = 0\,. \tag{5.16}$$

The Einstein equations decouple and are solved by the AdS Schwarzschild black hole*

$$e^{-\alpha^{(0)}} \equiv h = 1 - \left(\frac{r_+}{r}\right)^3\,, \qquad \beta^{(0)} = \gamma^{(0)} = 0\,. \tag{5.17}$$

To solve the Maxwell equations, it is convenient to introduce the coordinate

$$z = \frac{r_+}{r}\,, \tag{5.18}$$

so that the boundary is at $z = 0$ and the horizon at $z = 1$. Writing the $U(1)$ potential in terms of Fourier modes

$$A_t^{(0)} = \mu(1-\delta)\mathcal{A}_0(z) + \mu\delta\mathcal{A}_1(z)\cos Qx\,, \tag{5.19}$$

*There are also other inhomogeneous black hole spacetimes obtained by perturbing Reissner-Nordström black hole [?].

we deduce the mode equations

$$\mathcal{A}_n''(z) - \frac{n^2 Q^2}{r_+^2 h(z)} \mathcal{A}_n(z) = 0 \quad (n = 0, 1) \ , \tag{5.20}$$

to be solved together with the boundary conditions $\mathcal{A}_n(0) = 1$, $\mathcal{A}_n(1) = 0$. Here, $h(z) = 1 - z^3$ (Eq. (5.17)) and $'$ denotes a derivative with respect to z. For $n = 0$, we obtain

$$\mathcal{A}_0(z) = 1 - z \, . \tag{5.21}$$

For $n = 1$, a good analytic approximation to the solution is given by [40, 43]

$$\mathcal{A}_1(z) \approx \frac{\sinh \left[\frac{Q}{r_+}(1-z) \right]}{\sinh \frac{Q}{r_+}} \, . \tag{5.22}$$

The error vanishes at both ends ($z = 0, 1$) and attains a maximum value at an intermediate z. As $Q \to 0$, this maximum value decreases like Q^2, whereas as $Q \to \infty$, it decays exponentially. Numerically, for $Q/r_+ \sim 0.1, 1, 10$, we obtain a maximum error of $10^{-4}, 0.01, 0.001$, respectively.

Let us define

$$\Psi = \frac{\langle \mathcal{O}_\Delta \rangle^{(0)}}{\sqrt{2}} \frac{z^\Delta}{r_+^\Delta} \sum_{n \geq 0} F^{(n)}(z) \, \cos nQx, \tag{5.23}$$

with $F^{(0)}(z = 0) = 1$, such that the expectation value of the condensate is

$$\langle \mathcal{O}_\Delta \rangle = \sum_{n \geq 0} \langle \mathcal{O}_\Delta \rangle^{(n)} \cos nQx, \tag{5.24}$$

with

$$\langle \mathcal{O}_\Delta \rangle^{(n)} = \langle \mathcal{O}_\Delta \rangle^{(0)} \, F^{(n)}(z = 0). \tag{5.25}$$

Here, $\Delta = \Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2}$ is the dimension of the condensate operator $\mathcal{O}_\Delta$. We shall examine the range

$$\frac{1}{2} < \Delta < 3, \tag{5.26}$$

where $\Delta > \frac{3}{2}$ $\left(\Delta < \frac{3}{2}\right)$ for $\Delta = \Delta_+$ $(\Delta = \Delta_-)$, corresponding to masses in the range $0 > m^2 \geq -\frac{9}{4}$.

To gain more insight, let us solve the equation of motion for $F^{(0)}$ by treating the electrostatic potential as a perturbation to a leading order solution, which is nothing but that of a scalar field on an AdS Schwarzschild black hole without any background $U(1)$ gauge field

$$F_0^{(0)\prime\prime} + \left(\frac{h'}{h} + \frac{2(\Delta - 1)}{z}\right) F_0^{(0)\prime} - \frac{\Delta^2 z}{h} F_0^{(0)} = 0. \tag{5.27}$$

Here, $F_0^{(0)}$ is the leading term of the solution. Strictly speaking, this perturbation is valid only when $\delta \approx 1$, however, as we shall see in a bit, the result is a good approximation even when δ is far away from unity. The solution that satisfies the correct boundary conditions at $z = 0$ is

$$F_0^{(0)} = {}_2F_1\left(\frac{\Delta}{3}, \frac{\Delta}{3}; \frac{2\Delta}{3}; z^3\right), \tag{5.28}$$

where ${}_2F_1$ is the Gauss hypergeometric function. There is another solution, which corresponds to the solution with the correct boundary conditions for $\Delta \to 3 - \Delta$

$$\tilde{F}_0^{(0)} = z^{3-2\Delta}\, {}_2F_1\left(\frac{3-\Delta}{3}, \frac{3-\Delta}{3}; \frac{2(3-\Delta)}{3}; z^3\right). \tag{5.29}$$

Using perturbation theory, we obtain the next leading order solution, which is given by

$$
\begin{aligned}
F_1^{(0)}(z) &= \frac{\mu_c^2}{3-2\Delta}\left(F_0^{(0)}(z)\int_0^z \frac{dz'}{z'^{2-2\Delta}}\tilde{F}_0^{(0)}\,\frac{\mathcal{A}}{h}\,F_0^{(0)} - \tilde{F}_0^{(0)}(z)\int_0^z \frac{dz'}{z'^{2-2\Delta}}F_0^{(0)}\,\frac{\mathcal{A}}{h}\,F_0^{(0)}\right) \\
&\equiv \frac{\mu_c^2}{3-2\Delta}\left(F_0^{(0)}(z)\,\tilde{a}(z) - \tilde{F}_0^{(0)}(z)\,a(z)\right), \qquad (5.30)
\end{aligned}
$$

where

$$
\mathcal{A}(z) = (1-\delta)^2\,{A^{(0)}}^2 + \frac{\delta^2\,{A^{(1)}}^2}{2}\ . \qquad (5.31)
$$

As both $F_0^{(0)}$ and $\tilde{F}_0^{(0)}$ diverge logarithmically at the horizon, we obtain the following singularity for the full solution

$$
\begin{aligned}
F^{(0)}(z\to 1) &= F_0^{(0)}(z\to 1) + F_1^{(0)}(z\to 1) + \cdots \\
&\approx \log(1-z)\left\{\frac{\Gamma\left(\frac{2\Delta}{3}\right)}{\Gamma^2\left(\frac{\Delta}{3}\right)}\left[1+\frac{\mu_c^2\,\tilde{a}(1)}{3-2\Delta}\right] - \frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)}\frac{\mu_c^2\,a(1)}{3-2\Delta}\right\}, \qquad (5.32)
\end{aligned}
$$

where

$$
\begin{aligned}
a(1) &\approx (1-\delta)^2\,a_0^{(0)} + \frac{\Gamma(2\Delta-1)}{2^{2\Delta}}\,\frac{\delta^2}{Q_c^{2\Delta-1}}, \\
\tilde{a}(1) &\approx (1-\delta)^2\,\tilde{a}_0^{(0)} + \frac{1}{8}\,\frac{\delta^2}{Q_c^2}, \qquad (5.33)
\end{aligned}
$$

with

$$
\begin{aligned}
a_0^{(0)} &= \int_0^1 \frac{dz}{z^{2-2\Delta}}F_0^{(0)}\,\frac{(1-z)^2}{h}\,F_0^{(0)}, \\
\tilde{a}_0^{(0)} &= \int_0^1 \frac{dz}{z^{2-2\Delta}}\tilde{F}_0^{(0)}\,\frac{(1-z)^2}{h}\,F_0^{(0)}. \qquad (5.34)
\end{aligned}
$$

In obtaining Eq. (5.33), we have approximated the subleading integrals by evaluating the integrand near the boundary $z=0$.

Requiring regularity at the horizon, we obtain

$$\begin{aligned}\frac{1}{\mu_c^2} &= \frac{1}{3-2\Delta}\left(\frac{\Gamma^2\left(\frac{\Delta}{3}\right)}{\Gamma\left(\frac{2\Delta}{3}\right)}\frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)}a(1)-\tilde{a}(1)\right)\\ &= \left(\frac{\Gamma^2\left(\frac{\Delta}{3}\right)}{\Gamma\left(\frac{2\Delta}{3}\right)}\frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)}a_0^{(0)}-\tilde{a}_0^{(0)}\right)\frac{(1-\delta)^2}{3-2\Delta}+\frac{\Gamma^2\left(\frac{\Delta}{3}\right)}{\Gamma\left(\frac{2\Delta}{3}\right)}\frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)}\frac{\Gamma(2\Delta-1)}{2^{2\Delta}(3-2\Delta)}\frac{\delta^2}{Q_c^{2\Delta-1}},\end{aligned} \tag{5.35}$$

where we have dropped the term proportional to $1/Q_c^2$ on the last line. Both terms in Eq. (5.35) are small, but we make two different approximations in which one is a lot larger than the other. When the first term is significantly larger than the second one, for $\Delta = 1$, we have

$$\mu_c^2 = \frac{1.19}{(1-\delta)^2} - \frac{0.91\,\delta^2}{(1-\delta)^4\,Q_c}, \tag{5.36}$$

When the second term is significantly larger than the first, we have

$$\mu_c^2 = \frac{\Gamma\left(\frac{2\Delta}{3}\right)}{\Gamma^2\left(\frac{\Delta}{3}\right)}\frac{\Gamma^2\left(\frac{3-\Delta}{3}\right)}{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}\frac{2^{2\Delta}(3-2\Delta)}{\Gamma(2\Delta-1)}\frac{Q_c^{2\Delta-1}}{\delta^2}. \tag{5.37}$$

For $\Delta = 1$, $\delta = 1$, we then have $\mu_c^2 = 1.55\,Q_c$ and

$$\frac{T_c}{\mu} = \frac{0.15}{Q/\mu}, \tag{5.38}$$

We would like to note that for $\delta = 1$ and $\Delta > 3/2$, Eq. 5.37 does not have any solutions. This is related to the fact that for $\delta = 1$ and $\Delta > 3/2$, there is a critical Q^* such that $T_c(Q > Q^*) = 0$, as we have mentioned earlier.

Let us also comment on the $\Delta = 1/2$ unitarity limit, which is singular. To approach it, we introduce a cutoff Λ in Q-space. From Eq. (5.35), we have

$$\frac{1}{\mu_c^2(Q)} - \frac{1}{\mu_c^2(\Lambda)} = \frac{\Gamma^2\left(\frac{\Delta}{3}\right)}{\Gamma\left(\frac{2\Delta}{3}\right)} \frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)} \frac{\Gamma(2\Delta-1)}{2^{2\Delta}(3-2\Delta)} \delta^2 \left(Q^{1-2\Delta} - \Lambda^{1-2\Delta}\right). \tag{5.39}$$

If $\Delta > 1/2$, we can safely take the limit $\Lambda \to \infty$, in which $\Lambda^{1-2\Delta} \to 0$ and $\mu_c(\Lambda)$ has a finite limit. This is not so for the case of $\Delta = 1/2$. Taking the limit $\Delta \to 1/2$, we have

$$\lim_{\Delta \to \frac{1}{2}} \left[\frac{1}{\mu_c^2(Q)} - \frac{1}{\mu_c^2(\Lambda)}\right] = \frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)} \frac{\delta^2}{4} \log\frac{\Lambda}{Q}, \tag{5.40}$$

which shows that the limit $\Lambda \to \infty$ is not well defined and that for $\Delta = 1/2$, the chemical potential is no longer a physical quantity. A well defined physical quantity would be

$$\frac{d}{d\log Q}\left[\frac{1}{\mu_c^2(Q)}\right] = -\frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma^2\left(\frac{5}{6}\right)} \frac{\delta^2}{4}. \tag{5.41}$$

5.3 Adding Fluctuations

With the choice of boundary conditions (5.9), the lowest-order stress-energy tensor $\mathcal{T}_{ab}^{(0)}$ has modes with $n \leq 2$ due to the fact that it is quadratic in the $U(1)$ potential. The same should be true for the first-order corrections to the metric. Explicitly, the non-vanishing components of the zeroth-order electromagnetic stress-energy tensor are

$$\begin{aligned}
\mathcal{T}^{(0)\,t}{}_{t} &= -\mathcal{T}^{(0)\,y}{}_{y} = -\frac{z^4}{4}\left[\frac{\mathcal{E}_x^2}{h} + \mathcal{E}_z^2\right], \\
\mathcal{T}^{(0)\,z}{}_{z} &= -\mathcal{T}^{(0)\,x}{}_{x} = \frac{z^4}{4}\left[\frac{\mathcal{E}_x^2}{h} - \mathcal{E}_z^2\right], \\
\mathcal{T}^{(0)\,x}{}_{z} &= \frac{1}{h}\mathcal{T}^{(0)\,z}{}_{x} = -\frac{z^4}{2\,h}\mathcal{E}_x\mathcal{E}_z\,,
\end{aligned} \tag{5.42}$$

given in terms of the components of the electric field

$$\begin{aligned} \mathcal{E}_x &= \frac{\delta Q}{r_+}\mathcal{A}_1 \sin Qx \,, \\ \mathcal{E}_z &= (1-\delta)\mathcal{A}_0'(z) + \delta \mathcal{A}_1'(z) \cos Qx \,. \end{aligned} \tag{5.43}$$

To solve the Einstein equations at first order,

$$R^{(1)\,a}{}_{b} = \mathcal{T}^{(0)\,a}{}_{b}\,, \tag{5.44}$$

we set

$$\begin{aligned} \alpha^{(1)} &= \alpha_0^{(1)}(z) + \alpha_1^{(1)}(z)\cos Qx + \alpha_2^{(1)}(z)\cos 2Qx \,, \\ \beta^{(1)} &= \beta_0^{(1)}(z) + \beta_1^{(1)}(z)\cos Qx + \beta_2^{(1)}(z)\cos 2Qx \,, \\ \gamma^{(1)} &= \gamma_0^{(1)}(z) + \gamma_1^{(1)}(z)\cos Qx + \gamma_2^{(1)}(z)\cos 2Qx \,. \end{aligned} \tag{5.45}$$

We obtain five non-vanishing components for each set of functions $\{\alpha_i^{(1)}, \beta_i^{(1)}, \gamma_i^{(1)}\}$, where $i = 0, 1, 2$. Of the five equations, only three are independent and can be solved analytically for the three corresponding metric functions. After some algebra, we obtain the following system of equations for the modes of the metric functions.

For the Fourier zero modes, we obtain

$$\begin{aligned}
\alpha_0^{(1)\prime} - \left(\frac{3}{z} - \frac{h'}{h}\right)\alpha_0^{(1)} - \frac{z}{2}\left(\frac{4}{z} - \frac{h'}{h}\right)\gamma_0^{(1)\prime} & \\
-\frac{z^3\left(\frac{Q^2}{r_+^2}\delta^2\,\mathcal{A}_1^2 - h\,(2(1-\delta)^2\mathcal{A}_0'^2 + \delta^2\mathcal{A}_1'^2)\right)}{8h^2} &= 0\,, \\
\beta_0^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h}\right)\beta_0^{(1)\prime} - \frac{Q^2 z^2\delta^2\mathcal{A}_1^2}{4r_+^2 h^2} &= 0\,, \\
\gamma_0^{(1)\prime\prime} + \frac{Q^2 z^2\delta^2\mathcal{A}_1^2}{4r_+^2 h^2} &= 0\,.
\end{aligned} \tag{5.46}$$

We solve these equations by requiring that the functions be regular at the horizon ($z=1$) and vanish sufficiently fast at the boundary ($z=0$). We obtain

$$\gamma_0^{(1)}(z) = -\frac{Q^2\delta^2}{4\,r_+^2}\int_0^z dz'\int_0^{z'} dz''\,\frac{(z'')^2\,\mathcal{A}_1^2}{h^2}\,, \tag{5.47}$$

$$\beta_0^{(1)}(z) = -\frac{Q^2\delta^2}{4\,r_+^2}\int_0^z dz'\,\frac{(z')^2}{h}\int_{z'}^1 dz''\,\frac{\mathcal{A}_1^2}{h}\,, \tag{5.48}$$

$$\alpha_0^{(1)}(z) = \frac{z^3}{8h}\int_z^1 \overline{\alpha}_0^{(1)}(z')\,dz'\,, \tag{5.49}$$

where

$$\overline{\alpha}_0^{(1)}(z) = 2(1-\delta)^2{\mathcal{A}_0'}^2 + \delta^2{\mathcal{A}_1'}^2 - \frac{Q^2}{r_+^2}\delta^2\frac{\mathcal{A}_1^2}{h} - \gamma_0^{(1)\prime}\frac{4h - zh'}{z^3}\,. \tag{5.50}$$

For the Fourier first modes, we obtain

$$
\begin{aligned}
\alpha_1^{(1)\prime} - \left(\frac{3}{z} - \frac{\frac{Q^2}{r_+^2}\, z + 2h'}{2h}\right)\alpha_1^{(1)} & \\
-\frac{z}{2}\left(\frac{4}{z} - \frac{h'}{h}\right)\gamma_1^{(1)\prime} + \frac{Q^2 z(\beta_1^{(1)} - \gamma_1^{(1)})}{2r_+^2 h} & \\
+\frac{z^3\delta(1-\delta)\mathcal{A}_0'\mathcal{A}_1'}{2h} &= 0\,, \\
\beta_1^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h}\right)\beta_1^{(1)\prime} &= 0\,, \\
\gamma_1^{(1)\prime\prime} - \frac{Q^2}{r_+^2\, h}\alpha_1^{(1)} &= 0\,.
\end{aligned}
\tag{5.51}
$$

The second equation readily yields

$$
\beta_1^{(1)}(z) = 0\,. \tag{5.52}
$$

By eliminating $\alpha_1^{(1)}$ between the other two equations, we obtain a third order differential equation for $\gamma_1^{(1)}$. Then the possible behavior of $\gamma_1^{(1)}$ at the horizon is found to be a linear combination of $1-z$, $(1-z)\ln(1-z)$, and $(1-z)^{1+Q^2/6r_+^2}$. We fix the three integration constants by demanding $\gamma_1^{(1)}(0)=0$, $\gamma_1^{(1)\prime}(0)=0$, and $\gamma_1^{(1)\prime\prime} \lesssim \mathcal{O}(1/(1-z))$ at the horizon ($z=1$). The second boundary condition, together with Eqs. (5.51), ensure $\alpha_1^{(1)} \sim z^3$ at the boundary. The third boundary condition is necessary for the existence of a well-defined temperature (surface gravity), resulting in $\alpha_1^{(1)}(1)=0$, on account of the third equation in (5.51).

Finally, for the Fourier second modes, we obtain

$$
\begin{aligned}
\alpha_2^{(1)\prime} - \left(\frac{3}{z} - \frac{2\frac{Q^2}{r_+^2} z + h'}{h} \right) \alpha_2^{(1)} & \\
-\frac{z}{2} \left(\frac{4}{z} - \frac{h'}{h} \right) \gamma_2^{(1)\prime} + \frac{2Q^2 z}{r_+^2 h} (\beta_2^{(1)} - \gamma_2^{(1)}) & \\
+ \frac{\delta^2 z^3 \left(\frac{Q^2}{r_+^2} \mathcal{A}_1^2 + h {\mathcal{A}_1'}^2 \right)}{8h^2} &= 0 \,, \\
\beta_2^{(1)\prime\prime} - \left(\frac{2}{z} - \frac{h'}{h} \right) \beta_2^{(1)\prime} + \frac{\delta^2 Q^2 z^2 \mathcal{A}_1^2}{4 r_+^2 h^2} &= 0 \,, \\
\gamma_2^{(1)\prime\prime} - \frac{Q^2 \left(16\, h\, \alpha_2^{(1)} + \delta^2 z^2 \mathcal{A}_1^2 \right)}{4 r_+^2 h^2} &= 0 \,. \qquad (5.53)
\end{aligned}
$$

The second equation yields

$$
\beta_2^{(1)}(z) = \frac{Q^2 \, \delta^2}{4 r_+^2} \int_0^z dz' \, \frac{(z')^2}{h} \int_{z'}^1 dz'' \, \frac{\mathcal{A}_1^2}{h} \,. \qquad (5.54)
$$

We note that $\beta_2^{(1)} = -\beta_0^{(1)}$. Eliminating $\alpha_2^{(1)}$ between the other two equations, we obtain, as before, a third-order differential equation for $\gamma_2^{(1)}$, from which we deduce the possible near horizon behavior, $1-z$, $(1-z)\ln(1-z)$, and $(1-z)^{1+2Q^2/3r_+^2}$. As before, we fix the three integration constants by demanding $\gamma_2^{(1)}(0) = 0$, $\gamma_2^{(1)\prime}(0) = 0$, and $\gamma_2^{(1)\prime\prime} \lesssim \mathcal{O}(1/(1-z))$ at the horizon ($z = 1$). The second boundary condition, together with Eqs. (5.53), ensure $\alpha_2^{(1)} \sim z^3$ at the boundary. The third boundary condition is necessary for the existence of a well-defined temperature (surface gravity), resulting in $\alpha_2^{(1)}(1) = 0$, on account of the third equation in (5.53).

The equations for the various modes can be solved numerically subject to the boundary conditions outlined above. We have plotted $\alpha_n^{(1)}$ ($n = 0, 1, 2$) in Fig. 5.2 for representative values of Q, whereas $\beta_n^{(1)}$ ($n = 0, 2$; it vanishes for $n = 1$) is plotted in Fig. 5.3, and $\gamma_n^{(1)}$ is plotted in Figs. 5.4, 5.5, and 5.6, for $n = 0, 1$ and 2, respectively.

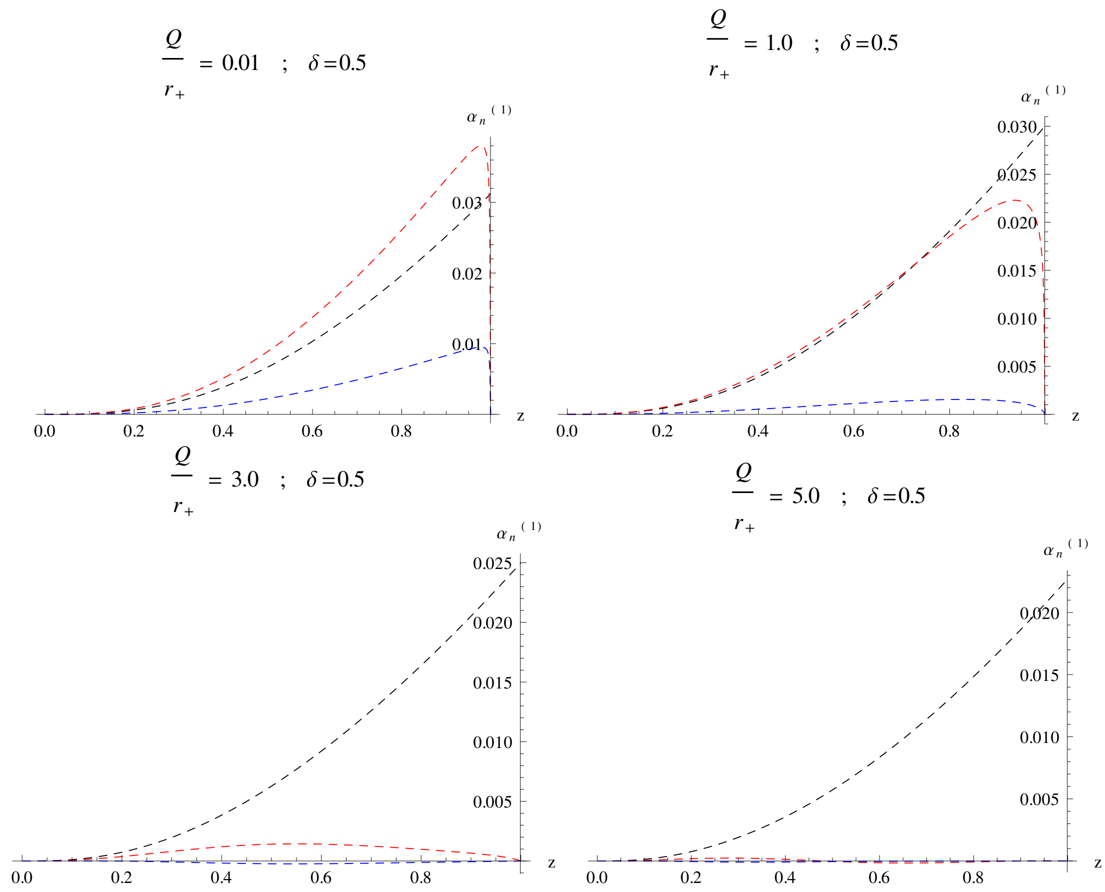

Figure 5.2: $\alpha_0^{(1)}$ (black), $\alpha_1^{(1)}$ (red) and $\alpha_2^{(1)}$ (blue) for $\delta = 0.5$, and $Q/r_+ = 0.1$, 1, 3 and 5.

Note that, the $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ components of metric perturbations are sourced by x-component of electric field (5.43), which vanishes at both small and large Q. The functions $\beta_n^{(1)}$s and $\gamma_n^{(1)}$s depend on Q via two terms: a direct proportionality factor Q^2 and area under the functions $\mathcal{A}_1^2/h$ or $z^2\mathcal{A}_1^2/h^2$. The first factor vanishes at $Q \to 0$, while the integrals vanish at $Q \to \infty$, due to $\mathcal{A}_1 \sim \frac{Q}{\sinh Q}(1-z)$ near the horizon.

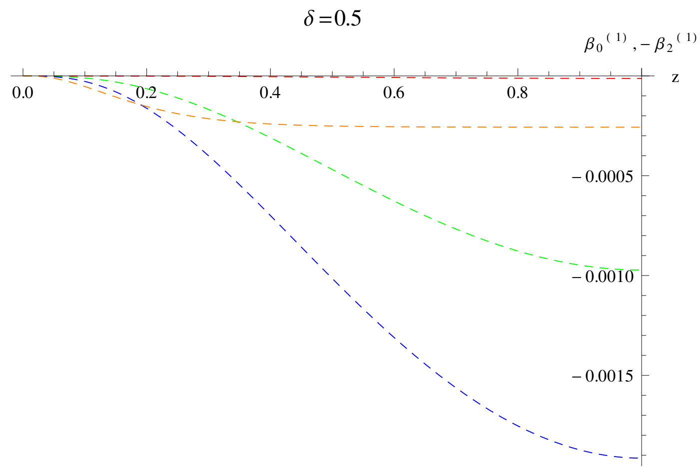

Figure 5.3: $\beta_0^{(1)} = -\beta_2^{(1)}$ for $\delta = 0.5$, and $QL^2/r_+ = 0.1$ (red), 1.0 (green), 2.0 (blue) and 8.0 (orange).

Consequently, $\beta_n^{(1)}$ and $\gamma_n^{(1)}$ $(n = 0, 1, 2)$ are very small in both limits $Q \ll r_+$ and $Q \gg r_+$. These functions are more significant in the intermediate range $2 < Q/r_+ < 3$ and decay rapidly on both sides, but even when they reach their maximum, they remain well below unity (see Figs. 5.3–5.6). Thus, their contribution to physical quantities is negligible in the entire range of Q.

Next, we discuss the behavior of $\alpha_n^{(1)}$ $(n = 0, 1, 2)$ which are physically important because they determine the temperature. Indeed, the Hawking temperature at first order in perturbation is

$$T = \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2}{r_+^2}\alpha^{(1)}(1)\right] . \tag{5.55}$$

Since $\alpha_n^{(1)}(1) = 0$ for $n \geq 1$, we have

$$\begin{aligned} \alpha^{(1)}(1) &= \alpha_0^{(1)}(1) = \frac{\overline{\alpha}_0(1)}{24} \\ &= \left.\frac{2(1-\delta)^2 + \delta^2 \mathcal{A}_1'^{\,2} - 3\gamma_0^{(1)\prime}}{24}\right|_{z=1} . \end{aligned} \tag{5.56}$$

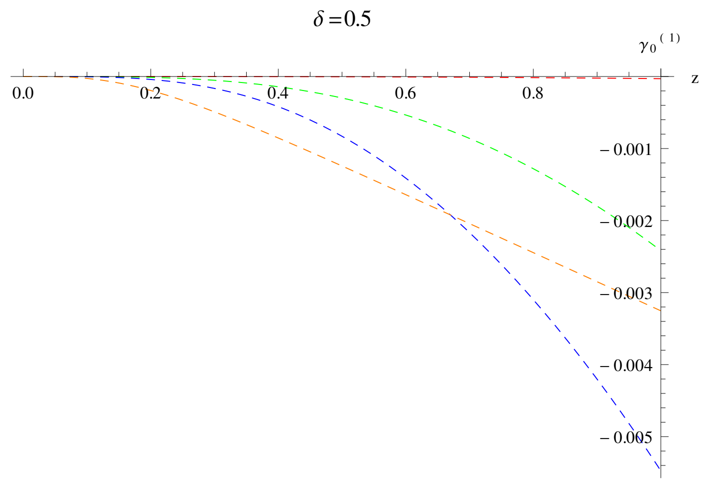

Figure 5.4: $\gamma_0^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green), 2.0 (blue) and 8.0 (orange).

We can calculate these functions analytically in the two important limits: $Q \to 0$ and $Q \to \infty$. In the limit $Q \to 0$, we obtain the analytic expressions

$$\begin{aligned} \alpha_0^{(1)} &= \left((1-\delta)^2 + \frac{\delta^2}{2}\right) \frac{z^3}{4\,(1+z+z^2)} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) , \\ \alpha_1^{(1)} &= \frac{(1-\delta)\,\delta\, z^3}{2\,(1+z+z^2)} (1-z)^{Q^2/6r_+^2} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) , \\ \alpha_2^{(1)} &= \frac{\delta^2\, z^3}{8\,(1+z+z^2)} (1-z)^{2Q^2/3r_+^2} + \mathcal{O}\left(\frac{Q^2}{r_+^2}\right) . \end{aligned} \tag{5.57}$$

At $Q = 0$ (or equivalently, $\delta = 0$), we recover the exact Reissner-Nordström solution representing the homogeneous system

$$\begin{aligned} e^{-\alpha} &= e^{-\alpha^{(0)}} \left(1 + \frac{\mu^2}{r_+^2}\,\alpha^{(1)}\right) \\ &= 1 - \left(1 + \frac{\mu^2}{4r_+^2}\right) z^3 + \frac{\mu^2}{4r_+^2} z^4 , \end{aligned} \tag{5.58}$$

where we used $\alpha^{(1)} = \alpha_0^{(1)} + \alpha_1^{(1)} + \alpha_2^{(1)}$ from Eq. (5.45) with $Q = 0$. We recover the Schwarzschild solution, which is the solution in the probe limit, in the limit $\mu \to 0$. As we increase μ, we move further away from the probe limit and the effects

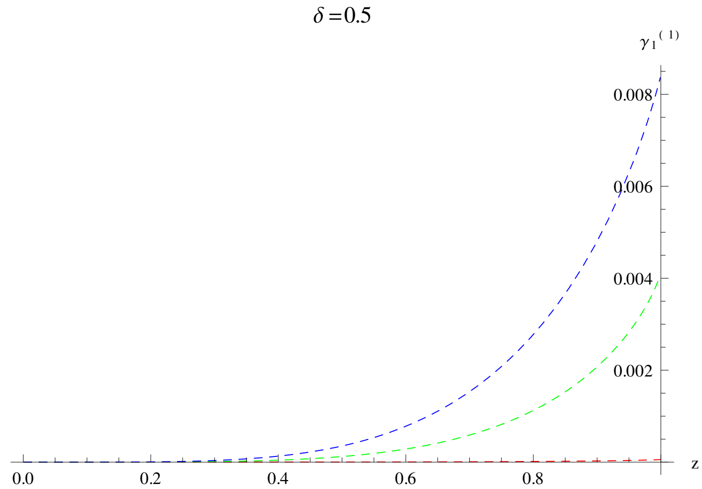

Figure 5.5: $\gamma_1^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green) and 2.0 (blue).

of back reaction to the metric become more pronounced. We reach extremality at $\mu/r_+ = 2\sqrt{3}$, but this lies outside the regime of validity of our approximation (Eq. (5.13)).

It should be noted that the convergence to the homogeneous system is not uniform. At the horizon, $\alpha_n^{(1)}(1) \to 0$ for $n = 1, 2$, and therefore $\alpha^{(1)}$ does not converge to its homogeneous counterpart. In other words, the limits $Q \to 0$ and $z \to 1$ do not commute. It follows that there is a discontinuity in the temperature which depends on the behavior of $\alpha_n^{(1)}$ at the horizon. From Eq. (5.55) in the limit $Q \to 0$, we obtain

$$T \approx \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2\left((1-\delta)^2 + \delta^2/2\right)}{12r_+^2}\right], \tag{5.59}$$

which is valid for small Q. Comparing this result with the homogeneous case, which is recovered by setting $\delta = 0$, we obtain an *enhancement* in temperature upon turning on modulation

$$\frac{\Delta T}{T} = \frac{T}{T_{\delta=0}} - 1 \approx \frac{\mu^2}{12r_+^2}\delta\left(2 - \frac{3\delta}{2}\right), \tag{5.60}$$

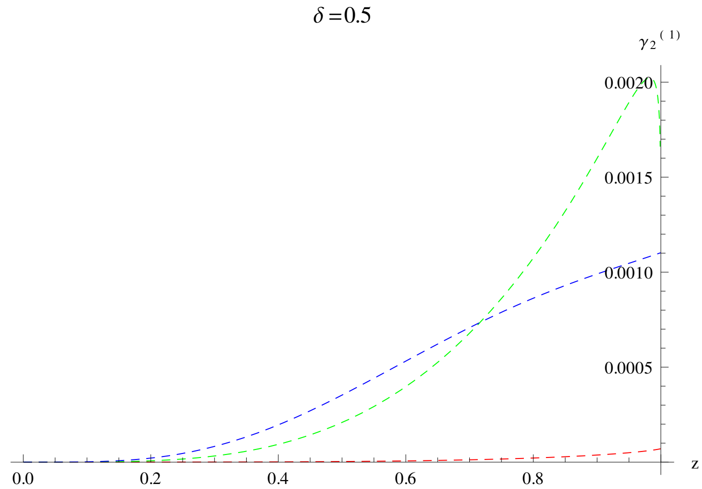

Figure 5.6: $\gamma_2^{(1)}$ for $\delta = 0.5$, and $Q/r_+ = 0.1$ (red), 1.0 (green) and 2.0 (blue).

with a maximum enhancement for $\delta = \frac{2}{3}$. The change in temperature is discontinuous, but this is an artifact of keeping only the first order in perturbation theory. This change in the temperature is expected to become smooth (yet remain steep) as higher orders in the perturbative expansion are included.

On the other hand, in the $Q \gg r_+$ regime, the contribution of $\mathcal{A}_1$ becomes exponentially small, and all functions except $\alpha_0^{(1)}$ become negligible. In this regime, we have

$$\alpha_0^{(1)} \approx \frac{(1-\delta)^2 \, z^3}{4\,(1+z+z^2)}\,. \tag{5.61}$$

So in the $Q \to \infty$ limit, we recover another exact Reissner-Nordström solution, albeit with less charge density,

$$e^{-\alpha} \approx 1 - \left(1 + \frac{\mu^2(1-\delta)^2}{4r_+^2}\right) z^3 + \frac{\mu^2(1-\delta)^2}{4r_+^2} z^4\,. \tag{5.62}$$

This coincides with the homogeneous solution (5.58) if $\delta = 0$, as expected.

We then deduce the temperature for large Q to be given by

$$T \approx \frac{3r_+}{4\pi}\left[1 - \frac{\mu^2(1-\delta)^2}{12r_+^2}\right] . \tag{5.63}$$

The Klein-Gordon equation for a static scalar field $\psi(z,x)$ of mass m and charge q reads

$$\sum_{i=z,x} \frac{1}{\sqrt{-g}}\partial_i\left(\sqrt{-g}g^{ii}\partial_i\psi\right) + \left(q^2 A_t^2 - m^2\right)\psi = 0 . \tag{5.64}$$

The mass is related to the conformal dimension Δ of the superconducting order parameter by

$$m^2 = \Delta\ (\Delta - 3)\ . \tag{5.65}$$

We have $\psi \sim z^\Delta$ as $z \to 0$. For a given set of parameters $\{\Delta, \delta, q, Q, \mu\}$, the wave equation yields the critical value of the radius of the horizon,

$$r_+ = r_{+c}\ . \tag{5.66}$$

It is convenient to define the eigenvalue

$$\lambda = \frac{q\mu}{r_{+c}} . \tag{5.67}$$

The critical temperature is then found from (5.55) by setting $r_+ = r_{+c}$. We obtain

$$\frac{T_c}{q\mu} = \frac{3}{4\pi}\left[\frac{1}{\lambda} - \frac{\lambda}{q^2}\alpha^{(1)}(1) + \mathcal{O}\left(\frac{1}{q^4}\right)\right] . \tag{5.68}$$

To simplify the wave equation, we note that the electrostatic potential has Fourier modes with wavenumbers nQ, where $n = 0, 1$, whereas the metric consists of modes with $n = 0, 1, 2$. It follows that ψ can be expanded in a Fourier series,

$$\psi(z,x) = z^{\Delta} F(z,x) \quad , \qquad F(z,x) = \sum F_n(z) \cos nQx \ , \tag{5.69}$$

where we factored out z^{Δ}, so that the modes $F_n(z) \sim$ const., as $z \to 0$. Using (5.69), the wave equation (5.64) can be written as an infinite system of coupled ordinary differential equations.

In the large Q regime, the higher modes become negligible, and the wave equation can be well approximated by the equation obeyed by the zero mode where all other modes have been set to zero,

$$F_0'' + \left[\frac{2(\Delta-1)}{z} + \frac{h_0'}{h_0}\right] F_0' + \frac{\Delta\left[(\Delta-3)(h_0-1) + z h_0'\right]}{z^2 h_0} F_0 \\ + \lambda^2 (1-\delta)^2 \frac{(1-z)^2}{h_0^2} F_0 = 0 \, , \tag{5.70}$$

where

$$h_0 \equiv e^{-\alpha^{(0)} - \frac{\mu^2}{r_+^2}\alpha_0^{(1)}} \approx h\left[1 - \frac{\mu^2}{r_+^2}\alpha_0^{(1)}\right] . \tag{5.71}$$

Expanding the scalar field, as we did with the other fields,

$$F = F^{(0)} + \left(\frac{\mu}{r_+}\right)^2 F^{(1)} + \dots \ , \tag{5.72}$$

where $F^{(0)}$ is the scalar field in the probe limit, we obtain for each Fourier mode,

$$F_n = F_n^{(0)} + \left(\frac{\mu}{r_+}\right)^2 F_n^{(1)} + \dots \ . \tag{5.73}$$

We also need to expand the eigenvalue (5.67) similarly,

$$\lambda = \lambda_0 + \left(\frac{\mu}{r_+}\right)^2 \lambda_1 + \dots \ . \tag{5.74}$$

We deduce for the probe limit zero mode,

$$F_0^{(0)\prime\prime} + \left[\frac{2(\Delta-1)}{z} + \frac{h'}{h}\right] F_0^{(0)\prime} + \frac{\Delta\left[(\Delta-3)(h-1)+zh'\right]}{z^2 h} F_0^{(0)} + \lambda_0^2 (1-\delta)^2 \frac{(1-z)^2}{h^2} F_0^{(0)} = 0\,. \quad (5.75)$$

which is the same as the equation for a homogeneous system in the probe limit, but with μ reduced to $\mu(1-\delta)$. The correction to the zeroth-order eigenvalue can be found using standard first-order perturbation theory. After some algebra, we obtain

$$\lambda_1 = \frac{\int_0^1 dz\, z^{2(\Delta-1)} h F_0^{(0)} \mathcal{H} F_0^{(0)}}{2\lambda_0 (1-\delta)^2 \int_0^1 dz\, z^{2(\Delta-1)} \frac{(1-z)^2}{h} [F_0^{(0)}]^2}\,, \quad (5.76)$$

where

$$\mathcal{H}\mathcal{F} \equiv \alpha_0^{(1)\prime} \mathcal{F}' + \left[\frac{\Delta(\Delta-3)}{z^2 h}\alpha_0^{(1)} + \frac{\Delta}{z}\alpha_0^{(1)\prime} - 2\lambda_0^2 (1-\delta)^2 \frac{(1-z)^2}{h^2}\alpha_0^{(1)}\right]\mathcal{F}\,. \quad (5.77)$$

The above results are valid in the $Q \to \infty$ limit. From (5.68), we deduce the asymptotic value of the temperature in this limit. As we decrease Q, an increasing number of Fourier modes become significant and one needs to solve a coupled system of ordinary differential equations of increasing complexity. This can be done numerically. The error in the numerical analysis can be reduced to the desired accuracy by including enough higher modes of the Fourier expansion. As $Q \to 0$, all modes become significant. In this limit, numerical methods based on a Fourier expansion become cumbersome. Fortunately, we can obtain analytic results in the limit $Q \to 0$, because all functions are slowly varying functions of x, and therefore the x-dependence can be ignored. We deduce the wave equation in the limit $Q \to 0$,

$$F'' + \left[\frac{2(\Delta-1)}{z} + \frac{\overline{h}'}{\overline{h}}\right] F' + \frac{\Delta}{z^2 \overline{h}} \left[(\Delta-3)(\overline{h}-1) + z\,\overline{h}'\right] F + \lambda^2 \frac{(1-z)^2}{\overline{h}^2} F = 0\,, \quad (5.78)$$

where

$$\overline{h} \equiv h\left[1 - \frac{\mu^2}{r_+^2}\overline{\alpha}\right] \quad , \qquad \overline{\alpha} = \alpha_0^{(1)} + \alpha_1^{(1)} + \alpha_2^{(1)}\,. \quad (5.79)$$

At zeroth order, this reduces to the probe limit result of the homogeneous case

$$F^{(0)''} + \left[\frac{2(\Delta-1)}{z} + \frac{h'}{h}\right] F^{(0)'} + \frac{\Delta}{z^2 h}\left[(\Delta-3)(h-1) + zh'\right] F^{(0)} + \lambda_0^2 \frac{(1-z)^2}{h^2} F^{(0)} = 0\,, \quad (5.80)$$

to be compared with the $Q \to \infty$ result (5.75). At first order, we obtain the correction to the eigenvalue in the limit $Q \to 0$,

$$\lambda_1 = \frac{\int_0^1 dz\, z^{2(\Delta-1)} h F_0^{(0)} \overline{\mathcal{H}} F_0^{(0)}}{2\lambda_0 (1-\delta)^2 \int_0^1 dz\, z^{2(\Delta-1)} \frac{(1-z)^2}{h} [F_0^{(0)}]^2}\,, \quad (5.81)$$

where

$$\overline{\mathcal{H}}\mathcal{F} \equiv \overline{\alpha}'\mathcal{F}' + \left[\frac{\Delta(\Delta-3)}{z^2 h}\overline{\alpha} + \frac{\Delta}{z}\overline{\alpha}' - 2\lambda_0^2 \frac{(1-z)^2}{h^2}\overline{\alpha}\right]\mathcal{F}\,. \quad (5.82)$$

From (5.68), we deduce the value of the temperature in the limit $Q \to 0$. At small Q, we obtain from Eq. (5.60) the enhancement in the critical temperature,

$$\frac{\Delta T_c}{T_c} \approx \frac{\lambda^2}{12 q^2}\delta\left(2 - \frac{3\delta}{2}\right) + \mathcal{O}\left(\frac{1}{q^2}\right)\,, \quad (5.83)$$

which vanishes at the probe limit ($q \to \infty$) and becomes significant away from it. However, we stress that the above results are not accurate in the small q limit, as they are only first-order $\mathcal{O}(1/q^2)$ results.

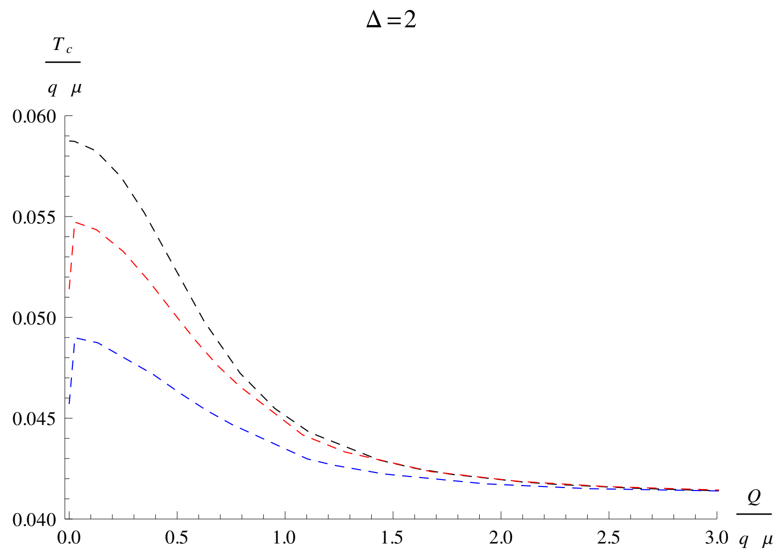

Figure 5.7: From top to bottom: T_c *vs.* Q for $\Delta = 2$ and $(\delta, q^2) = (0.3, \infty)$, $(0.2, 7.92)$ and $(0.1, 4.22)$. Parameters are chosen so that curves asymptote to $T_c/(q\mu) = 0.041$ as $Q \to \infty$.

The wave equation is solved numerically subject to the boundary conditions $F_0 \sim z^\Delta$ at the boundary and the demand of regularity at the horizon ($F_0(1) < \infty$). The results are shown in Figs. 5.7 and 5.8, for $\Delta = 2$ and 3, respectively. In each case, we have chosen the other parameters so that the curves asymptote to the same temperature as $Q \to \infty$. We note that all curves exhibit a jump at $Q = 0^+$, showing the enhancement of the critical temperature once modulation is switched on, in agreement with our analytic result (5.83). As Q increases, the critical temperature decreases monotonically. The jump vanishes in the probe limit which is obtained for $\mu = 0$ (Schwarzschild black hole). For any given Q, the critical temperature attains its maximum value at this limit. Put differently, back reaction to the metric lowers the critical temperature. Correspondingly, in the dual boundary system, quantum fluctuations result in a reduction in the critical temperature for a given modulation

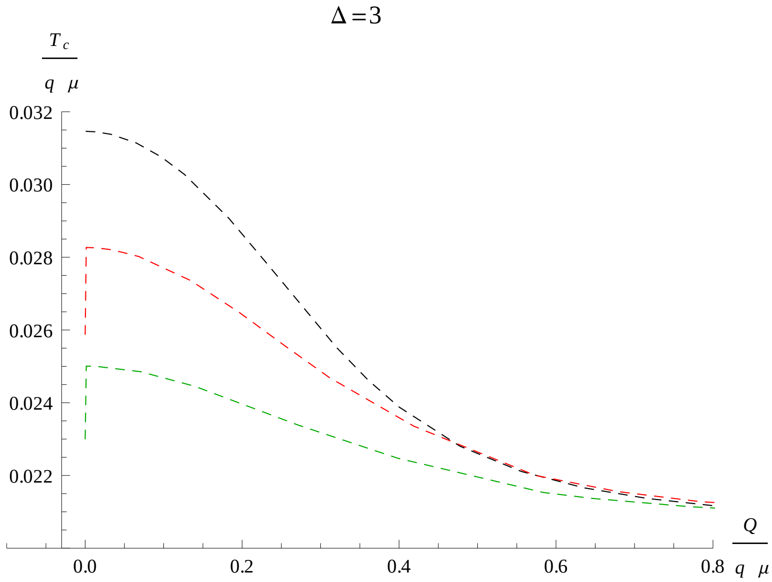

Figure 5.8: From top to bottom: T_c *vs.* Q for $\Delta = 3$ and $(\delta, q^2) = (0.34, \infty)$, $(0.2, 20.65)$ and $(0.1, 13.13)$. Parameters are chosen so that curves asymptote to $T_c/(q\mu) = 0.020$ as $Q \to \infty$.

vector Q.

5.4 Superconducting dome

With growing evidences of a monotonic relation between doping and modulation [54, 55, 56, 57], it is plausible that the qualitative behavior of the critical temperature viewed either as a function of doping or as a function of the modulation wave number should be similar. If this is true, then doping temperature phase diagram should be qualitatively similar to modulation temperature phase diagram. Therefore, as a step toward realizing a realistic holographic model of cuprate, it is crucial to show that indeed there exists a regime in the parameter space in which the holographic striped superconductor exhibits a superconducting dome.

Below we show that there is a critical modulation wave number Q_*, above which the critical temperature vanishes. The value Q_* can be estimated analytically by considering the large modulation wave number regime where $Q \gg T_c$. In this regime, we can see from the numeric that the effects of backreaction are suppressed. Furthermore, the higher modes are suppressed. Using the perturbative method of Ref. [43], we have

$$\frac{r_{+c}^2}{q^2\mu^2} = \frac{1}{2\Delta - 3}\left(\tilde{a}_c(1) - \frac{\Gamma^2\left(\frac{\Delta}{3}\right)}{\Gamma\left(\frac{2\Delta}{3}\right)}\frac{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}{\Gamma^2\left(\frac{3-\Delta}{3}\right)}a_c(1)\right), \tag{5.84}$$

where

$$\begin{aligned} a_c(1) &= \frac{1}{2^{2\Delta}}\frac{r_{+c}^{2\Delta-1}}{Q^{2\Delta-1}}\Gamma(2\Delta-1), \\ \tilde{a}_c(1) &= \frac{r_{+c}^2}{8Q^2}. \end{aligned} \tag{5.85}$$

Here, r_{+c} is the value of r_+ at the critical temperature. Therefore,

$$\begin{aligned} \frac{r_{+c}^{2\Delta-3}}{(q\mu)^{2\Delta-3}} &= \frac{2^{2\Delta}(2\Delta-3)}{\Gamma(2\Delta-1)}\frac{\Gamma\left(\frac{2\Delta}{3}\right)}{\Gamma^2\left(\frac{\Delta}{3}\right)}\frac{\Gamma^2\left(\frac{3-\Delta}{3}\right)}{\Gamma\left(\frac{2(3-\Delta)}{3}\right)}\frac{Q^{2\Delta-1}}{(q\mu)^{2\Delta-1}} \\ &\quad\times\left(\frac{1}{8(2\Delta-3)}\frac{q^2\mu^2}{Q^2} - 1\right), \end{aligned} \tag{5.86}$$

which means above

$$\left(\frac{Q_*}{q\mu}\right)^2 = \frac{1}{8(2\Delta-3)}, \tag{5.87}$$

we have no instability and $T_c = 0$. We can improve upon this approximation by iteratively solving the equation of motion

$$\partial_{\tilde{z}}^2 F^{(0)} + \frac{2(\Delta-1)}{\tilde{z}}\partial_{\tilde{z}}F^{(0)} + \lambda e^{-\tilde{z}}F^{(0)} = 0, \tag{5.88}$$

where $\tilde{z} = zr_+/(2Q)$ and $\lambda = q^2\mu^2/(8Q_*^2)$. We would like to evaluate this in the interval $\tilde{z} \in [0, \infty)$, with the boundary condition $F^{(0)} \to 0$ as $\tilde{z} \to \infty$. This differential equation can be solved exactly for $\Delta = 2$, but for other values $3/2 < \Delta \leq 3$, we can estimate λ by iteration. To do so, let us rewrite 5.88 as an integral equation

$$\begin{aligned} F^{(0)}(\tilde{z}) &= 1 - \frac{\lambda}{2\Delta - 3} \int_0^{\tilde{z}} dw\, w\, e^{-w}\, F^{(0)}(w) \\ &\quad + \frac{\lambda}{(2\Delta - 3)\tilde{z}^{2\Delta-3}} \int_0^{\tilde{z}} dw\, w^{2\Delta-2} e^{-w}\, F^{(0)}(w). \end{aligned} \tag{5.89}$$

Solving this iteratively, we reproduce Eq. 5.87 at zeroth order while a first order correction results in

$$\begin{aligned} \left(\frac{Q_*}{q\mu}\right)^2 &= \frac{3 - 4\Delta}{32} + \frac{(\Delta - 1)(\Delta - 2)}{4} \times \\ &\quad \times \left[\psi\left(\frac{5}{2} - \Delta\right) - \psi(3 - \Delta) + \frac{2\pi}{\sin 2\pi\Delta}\right]. \end{aligned} \tag{5.90}$$

Here, ψ denotes the digamma function. Despite appearances, as we can see in Fig. 5.10, this is a smooth function of Δ in the interval $(3/2, 3)$ (the two expressions above have almost indistinguishable graphs). Furthermore, at $\Delta \to 3^-$, we have $\frac{Q_*}{q\mu} \approx 0.158$, in good agreement with our numerical results.

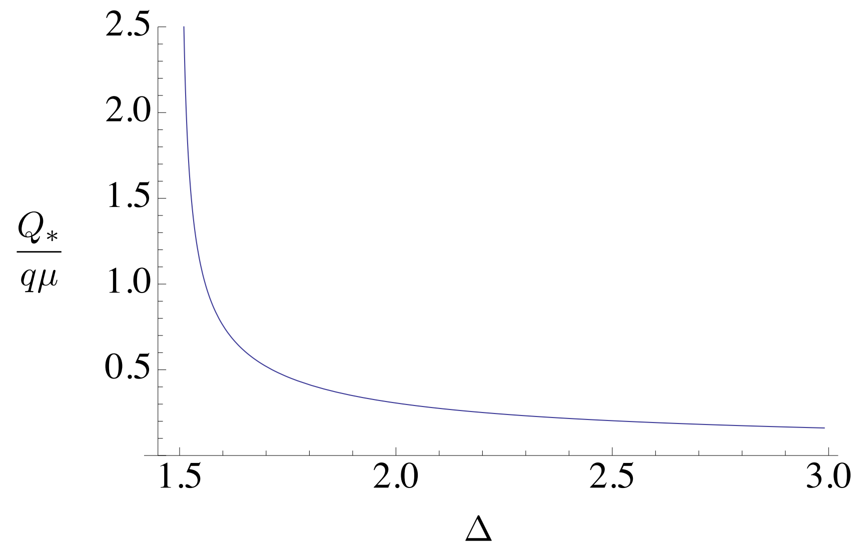

Figure 5.10: The end point of the superconducting dome as a function of the scaling dimension of the order parameter.

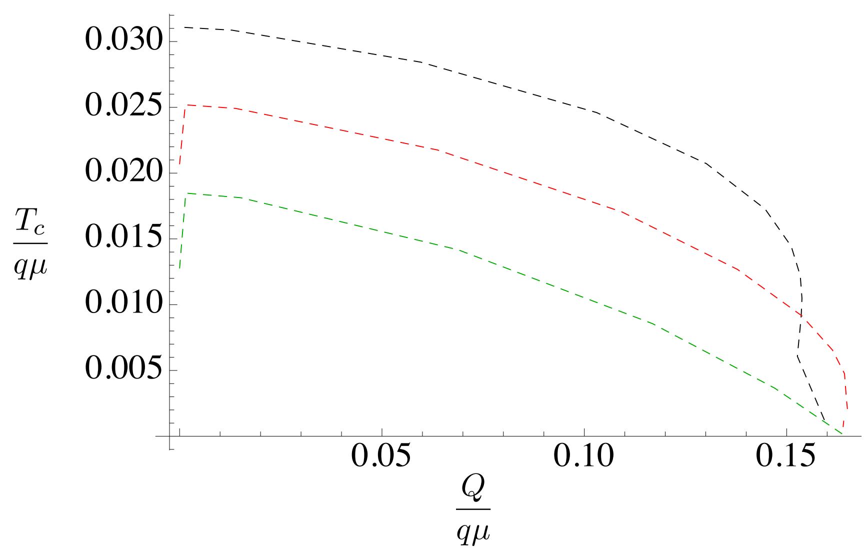

Figure 5.9: (Color online) The critical temperature as a function of modulation wavenumber. Here, $\Delta = 3$ and the black, red and green lines correspond to $q^2 = \infty$, 10 and 5, respectively.

By comparing the information concerning the end points of the dome, $Q = 0$ and $Q = Q_*$, with experiment, we can then in principle extract the value of Δ. This is

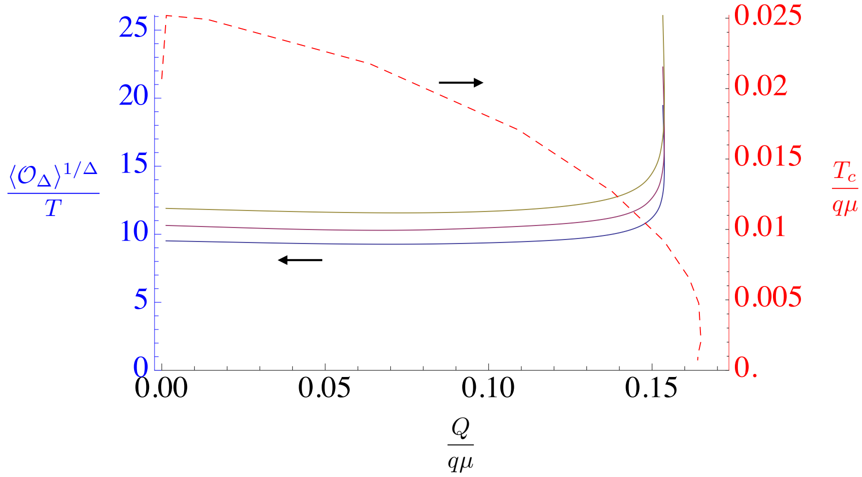

Figure 5.11: (Color online) The ratio of the gap with respect to temperature as a function of the wavenumber at fixed T/T_c. Here, $\Delta = 3$ and the blue, purple and yellow lines (bottom to top) correspond to $T/T_c = 0.975$, 0.95 and 0.9, respectively. For clarity, we have also shown the critical temperature, plotted in dashed line.

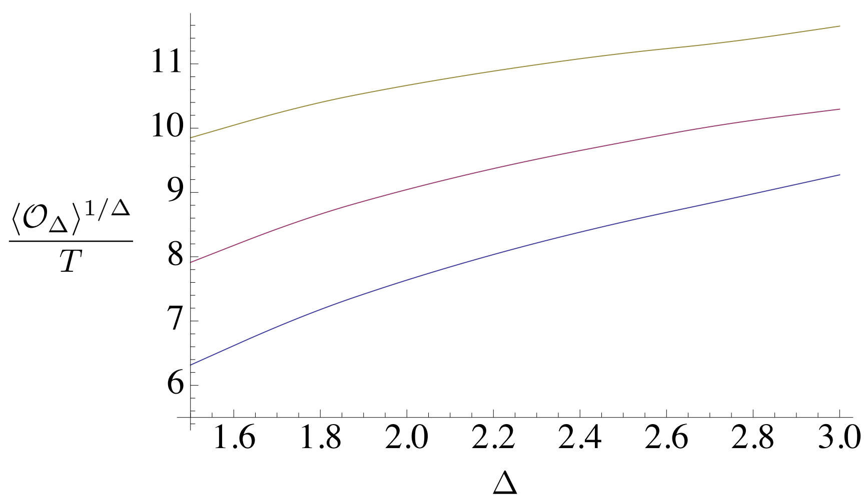

Figure 5.12: (Color Online) The ratio of the gap to temperature deep under the dome, as a function of the scaling dimension of the order parameter. Here, the blue, purple and yellow lines (bottom to top) correspond to $T/T_c = 0.975$, 0.95 and 0.9, respectively.

because the $Q = 0$ point gives us the information on q and upon substituting it into Eq. 5.90, we obtain Δ. Another method to extract the realistic value of Δ is by calculating the anisotropy of the optical conductivity akin to the calculation done in [45] (see also [46]) and comparing its scaling behavior with the observed behavior in cuprates [47].

Combining the results of [43] and [44], we expect that when the chemical potential averages to zero ($\delta = 1$) and the scaling dimension of the order parameter $3/2 < \Delta \leq 3$, the superconducting regime is capped at larger values of Q, while for sufficiently low q^2, it is capped at the smaller values of Q and thus, resulting in a superconducting dome. We show [58] that indeed this expectation is correct. Modulation dependence of condensate within this dome is found to be mild, except divergent near extrimality at overdoped regime.

We found that deep under the superconducting dome, the gap shows a mild dependence on Q. Interestingly, recent measurement of the gap at fixed temperature at low temperature also showed mild dependence on Q. We are hopeful that gap measurement near T_c from STM will be available to be compared with our prediction in Fig. 5.11 in the near future.

The ratio $\langle \mathcal{O}_\Delta \rangle^{1/\Delta}/T$ deep under the superconducting dome as a function of Δ is plotted in Fig. 5.12. The value is about 8-10, which is the same order of magnitude with the measured value of 29 [57]. We would like to note that the measurement is done at $T/T_c \approx 0.2$, which is beyond the regime of validity of our approximation. However, considering that in some holographic models, the ratio of gap over temperature can go as high as $\mathcal{O}(10^3)$, our result is encouraging.

Chapter 6

Conclusions

In this thesis we developed a phenomenological holographic model of strongly coupled $2+1$ dimensional striped superconductors and studied the interplay between charge density wave order and superconductivity. In particular, dependence of T_c on CDW wave vector Q is studied for condensates of various scaling dimension Δ while tuning the ratio δ of homogeneous and inhomogeneous part of charge density and modulation wave vector Q. The dynamics of fluctuations in such systems were analyzed in first order approximation and the effects of fluctuations on both T_c and condensate were studied. In the following section we summarize our major results.

6.1 Summary of Results and Outlook

First we analyzed the properties of a striped holographic superconductor in the probe limit at large modulation wavenumber Q, where the backreaction is negligible. We calculated the critical temperature T_c and the expectation value of the condensate $\langle \mathcal{O}_\Delta \rangle$ below T_c analytically for arbitrary values of the scaling dimension Δ. We found that in the absence of a homogeneous terms in the chemical potential, both T_c and $\langle \mathcal{O}_\Delta \rangle$ have a power law behavior for large Q for $\Delta < 3/2$. In particular, the critical temperature behaves as

$$T_c \propto Q^{-\frac{2\Delta-1}{3-2\Delta}}, \tag{6.1}$$

while the power of the condensate is such that the gap

$$\langle \mathcal{O}_\Delta \rangle^{1/\Delta} \propto Q. \tag{6.2}$$

On the other hand if the scaling dimension of the order parameter Δ is larger than 3/2 then there exist a critical modulation Q_*, beyond which no phase transition takes place. This critical modulation is given by,

$$\left(\frac{Q_*}{q\mu}\right)^2 = \frac{1}{8(2\Delta-3)} + \text{corrections}$$

We also found that the odd modes of the condensate vanish, while the higher even modes are suppressed

$$\frac{\langle \mathcal{O}_\Delta \rangle^{(n)}}{\langle \mathcal{O}_\Delta \rangle^{(0)}} \leq \mathcal{O}\left(\frac{\mu^{2n}}{Q^{2n}}\right). \tag{6.3}$$

In the case in which a homogeneous term is included in the chemical potential, both T_c and $\langle \mathcal{O}_\Delta \rangle$ approach constant values in the large Q limit, but the subleading terms are powers of Q. These constant values are the corresponding values for the homogeneous superconductors with chemical potential $\mu\delta$. We also found that the higher modes of the condensate are suppressed

$$\frac{\langle \mathcal{O}_\Delta \rangle^{(n)}}{\langle \mathcal{O}_\Delta \rangle^{(0)}} \leq \mathcal{O}\left(\frac{\mu^{2[n/2]}}{Q^{2[n/2]}}\right). \tag{6.4}$$

Next we incorporated fluctuations in system by including backreaction into the spcetime geometry. The two characteristic features of fluctuations in this system with modulated charge density seem to have a profound influence on T_c. First, fluctuations are found to be dominant in small modulation regime and exponentially suppressed at large modulation. This is expected, since for a CDW with modulation wave vector large compared to some characteristic energy scale of the system, the fluctuations are energetically costly are therefore suppressed. On the other hand for modulation small compared to energy scale of the system, the energy cost due modulated charge density is small and fluctuations play a dominant role in this regime. Second, fluctuations *compete* with superconductivity. As we turn on modulation, strength of fluctuations decrease monotonically causing a steep jump in T_c. After that, as we increase Q, the critical temperature decreases until it reaches the asymptotic value. In other words, we found an *enhancement* of the critical temperature due to inhomogeneity that comes from the stripe order.

$$\frac{\Delta T_c}{T_c} \approx \frac{\lambda^2}{12q^2}\delta\left(2-\frac{3\delta}{2}\right)+\mathcal{O}\left(\frac{1}{q^2}\right)\,, \tag{6.5}$$

which vanishes in the mean-field or probe limit ($q \to \infty$) and becomes significant away from it.

We have seen that the critical temperature vanishes asymptotically at Q_* the value of which is depends on the scaling dimension of the condensate. On the other hand at $Q = 0$ the critical temperature vanishes for sufficiently large q^2. In the regime of validity of the $1/q^2$ expansion, we are not able to crank up the fluctuations to reach vanishing critical temperature at $Q = 0$. However, as the exact solution at $Q = 0$ is known, we expect this to happen at $q^2 = 3/4 + \Delta(\Delta - 3)/2$ [59]. Consolidating the results we conclude that the holographic striped superconductor exhibits a superconducting dome with vanishing or nearly vanishing critical temperature at the end points. In other words, assuming a CDW normal state throughout the modulation range $0 \leq Q \leq Q_*$, the critical temperature of a holographic superconductor shows similar dependence on Q as critical temperature of cuprates depend on doping concentration x.

6.2 Future Work

The simple model considered in section 1 to study the interplay between stripe orders and superconductivity has one major limitation: anisotropy is introduced via externally modulated chemical potential. It is desired to extend this model into which both the superconducting order and stripe order emerge dynamically. Insight can be drawn from the existing holographic models that have been proposed to describe CDW [53]. One simple mechanism for dynamic emergence of CDW, is to introduce another $U(1)$ gauge field corresponding to an effective magnetic field acting on the spins in boundary theory. The scalar field is neutral under this gauge field.

In another approach towards a dynamic emergence of CDW in holographic system I am considering a system with higher powers of derivatives in Lagrangian. A similar non relativistic analysis of a classical dynamical systems [60] by Shapere et al. revealed motion of the system in their lowest energy state, forming a time analogue of crystalline spatial order.

In cuprates and iron pnictides the formation of Cooper pair can no longer be attributed to the electron-lattice coupling. Despite intensive studies, the correct pairing mechanism is still elusive. However, a common feature of these systems is a long range antiferromagnetic ground state in the non superconducting parent which, upon doping, evolves into loosely correlated magnetic excitations coexisting with superconductivity. To date, pairing mechanism mediated by magnetic fluctuations has been regarded as a leading candidate to resolve the problem of high-T_c superconductivity. Holographic models of pairing mechanism might be able to elucidate the underlying physics. In striped superconductors as we have seen, modulated pairing is possible as well, in which the cooper pair is spatially modulated.

$$\Phi^{\dagger}(\vec{R}) \sim cos(\vec{Q}_p \cdot \vec{R}) F^{\dagger}(\vec{R}) \tag{6.6}$$

Although most superconducting state has vanishing momentum $\vec{Q}_p = 0$, finite momentum pairing has been proposed by Fulde,Ferrel,Larkin and Ovchinnikov called the "FFLO" state. Such finite momentum condensate is related to "antiphase superconductivity" where neighbouring stripes are in opposite phase leading to strong suppression of bulk superconductivity. Holographic models of FFLO states have already been proposed. An interesting extension of present work will be to analyze modulation dependence of these states.

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Vita

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