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On a Quantum Form of the Binomial Coefficient

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I am submitting herewith a thesis written by Eric J. Jacob entitled "On a Quantum Form of the Binomial Coefficient." I have examined the final electronic copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, with a major in Mathematics.

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On a Quantum Form of the Binomial Coefficient

A Thesis Presented for the
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Degree
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Dedication

Dedicated to the memory of Dr. Boris Kupershmidt.

Acknowledgements

I would first most like to thank Dr. Boris Kupershmidt for his mathematical mentorship in my time as a student at the University of Tennessee Space Institute. He inspired me to study mathematics as it is beautiful in of itself. He acted as my initial advisor and introduced me to quantum number theory. He sadly passed away before this thesis was completed. I would like to thank Dr. K. C. Reddy for his assistance and his advice upon the completion of this thesis. He was willing to become my advisor after the passing of Dr. Kupershmidt. I would not have been able to complete this thesis without his help. Finally, I would like to thank the other members of my committee, Dr. Ken Kimble and Dr. Gary Flandro for their assistance, suggestions and advice.

Abstract

A unique form of the quantum binomial coefficient (n choose k) for k = 2 and 3 is presented in this thesis. An interesting double summation formula with floor function bounds is used for k = 3. The equations both show the discrete nature of the quantum form as the binomial coefficient is partitioned into specific quantum integers. The proof of these equations has been shown as well. The equations show that a general form of the quantum binomial coefficient with k summations appears to be feasible. This will be investigated in future work.

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Chapter 1

Introduction

The quantum form of a function, identity, theorem or expression is a generalization or expansion of the original form with the parameter q such that in the limit of $q \rightarrow 1^-$ the original form is retained[1]. Quantum numbers are referred to in the literature as q-analogs or q-extensions[1]. Quantum forms are applied in quantum groups[2-4] and have applications in a variety of fields such as quantum field theory[2-4].

Mathematical Introduction

The second quantized form of the binomial coefficient is investigated in this thesis. Quantized numbers of the first kind[1,5-8] are defined as,

$$[k]_q = \sum_{j=0}^{k-1} q^j = \frac{1-q^k}{1-q}$$

where $0 < q < 1$. The second quantized numbers[9-11] are given as,

$$[k]_{\tilde{q}} = \sum_{j=0}^{k-1} q^{2j-(k-1)} = \frac{q^k - q^{-k}}{q - q^{-1}}$$

where again, $0 < q < 1$. The second quantized form has a symmetrical quality as q and q^{-1} can be exchanged. The first and second quantum forms of three are shown below in Figure 1 as an example.

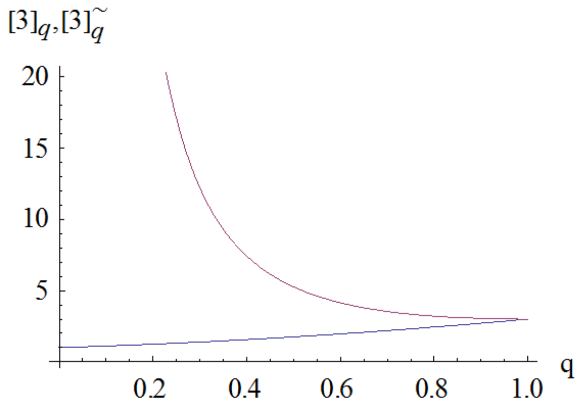

Figure 1. Evaluation of $[3]_q$ (blue) and $[3]^{\sim}_q$ (purple) quantized forms over $0 \leq q < 1$

The second quantum form is related to the first by,

$$[k]^{\sim}_q = \left(\frac{q + q^{1-k}}{1+q}\right)[k]_q$$

These quantum numbers converge to the classical solution, k, in the limit of $q \rightarrow 1^-$.

$$\lim_{q \rightarrow 1^-} [k] = \lim_{q \rightarrow 1^-} \frac{1-q^k}{1-q} = k$$

$$\lim_{q \rightarrow 1^-} [k]^{\sim}_q = \lim_{q \rightarrow 1^-} \frac{q^k - q^{-k}}{q - q^{-1}} = k$$

The definition of the classical binomial coefficient is,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

The first order q-binomial coefficient[12-13] is given by,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

The second quantized binomial coefficient[9-11] follows the same form,

$$\begin{bmatrix} n \\ k \end{bmatrix}^{\sim}_q = \frac{[n]!^{\sim}_q}{[k]!^{\sim}_q\,[n-k]!^{\sim}_q}$$

where,

$$[n]!_q^{\sim} = \prod_{i=1}^{n} [i]_q^{\sim}$$

and,

$$[1]_q^{\sim} = 1$$

$$[0]_q^{\sim} = 0$$

$$[0]!_q^{\sim} = 1$$

Interestingly, the quantum form of the binomial coefficient generates a series of discrete quantum integers. These values are calculated by expanding the factorial into q form and then algebraically manipulating them back into discrete quantized values. For example,

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q^{\sim} = \frac{[5]!_q^{\sim}}{[3]!_q^{\sim}\,[2]!_q^{\sim}} = \frac{[5]^{\sim}[4]^{\sim}[3]^{\sim}[2]^{\sim}}{[3]^{\sim}[2]^{\sim}[2]^{\sim}} = \frac{[5]^{\sim}[4]^{\sim}}{[2]^{\sim}} =$$

$$= \frac{(-\frac{1}{q^4} + q^4)(-\frac{1}{q^5} + q^5)}{(-\frac{1}{q} + q)(-\frac{1}{q^2} + q^2)} = 2 + \frac{1}{q^6} + \frac{1}{q^4} + \frac{2}{q^2} + 2q^2 + q^4 + q^6 = [7]^{\sim} + [3]^{\sim}$$

A symbolic mathematics code (Appendix A) was developed to automatically generate these discrete values for the second quantized form of the binomial coefficient. The set of discrete values are shown in Table 1. For simplicity the notation has been modified such that,

$$[k]^{\sim} \equiv [k]_q^{\sim}$$

Table 1. Examples of the Discretized Second Quantum Binomial Coefficient

n	k = 2	k = 3
2	$[1]^{\sim}$	-
3	$[3]^{\sim}$	$[1]^{\sim}$
4	$[5]^{\sim}+[1]^{\sim}$	$[4]^{\sim}$
5	$[7]^{\sim}+[3]^{\sim}$	$[7]^{\sim}+[3]^{\sim}$
6	$[9]^{\sim}+[5]^{\sim}+[1]^{\sim}$	$[10]^{\sim}+[6]^{\sim}+[4]^{\sim}$
7	$[11]^{\sim}+[7]^{\sim}+[3]^{\sim}$	$[13]^{\sim}+[9]^{\sim}+[7]^{\sim}+[5]^{\sim}+[1]^{\sim}$
8	$[13]^{\sim}+[9]^{\sim}+[5]^{\sim}+[1]^{\sim}$	$[16]^{\sim}+[12]^{\sim}+[10]^{\sim}+[8]^{\sim}+[6]^{\sim}+[4]^{\sim}$
9	$[15]^{\sim}+[11]^{\sim}+[7]^{\sim}+[3]^{\sim}$	$[19]^{\sim}+[15]^{\sim}+[13]^{\sim}+[11]^{\sim}+[9]^{\sim}+[7]^{\sim}+[7]^{\sim}+[3]^{\sim}$

It is important to note that the quantum form is generating a specific partitioning of the binomial coefficient. A similar portioning is present in the first quantum form of the binomial coefficient. As an example, the three forms of the binomial coefficient are given as,

$$\binom{6}{3} = 20$$

$$\begin{bmatrix}6\\3\end{bmatrix}_q = [10]_q + [8]_q + [7]_q - [3]_q - [2]_q$$

$$\begin{bmatrix}6\\3\end{bmatrix}_q^{\sim} = [10]_q^{\sim} + [6]_q^{\sim} + [4]_q^{\sim}$$

It is interesting to note that in the limit of $q \to 1^-$ the quantum integers revert to classical integers and the classical value of 20 is retained in both quantum cases. This thesis will focus on the symmetric second quantized form of the binomial coefficient and its partitioning.

Using tables of the second quantum discretized values a pattern was observed. All of the terms had a maximum value of $k(n - k) + 1$. For $k = 2$ the pattern is apparent as all terms are separated by a space of 4. This was represented as a summation and only determining the summations bounds was necessary. For $k = 3$ the pattern was more complex as there are irregular spacing's of 0, 2, 4 or 6. It will be shown later that this can be represented by a series of overlapping patterns with different summation bounds. Finally, a proof will be made to show that these equations are valid.

Chapter 2

[n choose 2]~ Equation

The set of values with k = 2 are easily given as,

$$\begin{bmatrix} n \\ k=2 \end{bmatrix}_q^{\sim} = \sum_{i=0}^{f} [\mathrm{k(n-k)} + 1 - 4i]_q^{\sim}$$

where,

$$f = \left\lfloor \frac{n-2}{2} \right\rfloor$$

Here the floor function[14] is defined as,

$$\lfloor x \rfloor = \max\{m \in Z | m \leq x\}$$

where *x* is a real number, *m* is an integer, and Z is the set of integers. Put more simply, the floor of x is the largest integer not greater than x.

Proof

The proof of this equation is given easily.

$$\begin{bmatrix} \mathrm{n} \\ 2 \end{bmatrix}_\mathrm{q}^{\sim} \overset{?}{=} \sum_{\mathrm{i}=0}^{\mathrm{f}} [2(\mathrm{n}-2) + 1 - 4\mathrm{i}]_\mathrm{q}^{\sim}$$

Expanding the right hand side (RHS) into q form yields,

$$\sum_{\mathrm{i}=0}^{\mathrm{f}} [2(\mathrm{n}-2) + 1 - 4\mathrm{i}]_\mathrm{q}^{\sim} = \left(\frac{1}{\mathrm{q}-\mathrm{q}^{-1}}\right) \sum_{\mathrm{i}=0}^{\mathrm{f}} \left(\mathrm{q}^{2\mathrm{n}-3-4\mathrm{i}} - \mathrm{q}^{-2\mathrm{n}+3+4\mathrm{i}}\right)$$

$$= \left(\frac{1}{\mathrm{q}-\mathrm{q}^{-1}}\right) \sum_{\mathrm{i}=0}^{\mathrm{f}} \left(\mathrm{q}^{2\mathrm{n}-3}\mathrm{q}^{-4\mathrm{i}} - \mathrm{q}^{-2\mathrm{n}+3}\mathrm{q}^{4\mathrm{i}}\right)$$

Geometric series are applied in order to eliminate the summations. Only the terms which are functions of i must be evaluated within the summations.

$$\sum_{i=0}^{f(n)} q^{-4i} = \left(\frac{1 - q^{-4(f(n)+1)}}{1 - q^{-4}}\right)$$

$$\sum_{i=0}^{f(n)} q^{4i} = \left(\frac{1 - q^{4(f(n)+1)}}{1 - q^{4}}\right)$$

The function f(n) is simplified by expanding n into n = 2N + m. Where m = 2 or 3. This is motivated by the fact that the floor function has a period of 2.

$$f(n) = \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lfloor \frac{2N + m - 2}{2} \right\rfloor = N + \left\lfloor \frac{m-2}{2} \right\rfloor = N$$

Expanding n within the full equation and applying the summation reductions results in,

$$\left(\frac{1}{q - q^{-1}}\right)\left\{q^{4N+2m-3}\left(\frac{1 - q^{-4(N+1)}}{1 - q^{-4}}\right) - q^{-4N-2m+3}\left(\frac{1 - q^{4(N+1)}}{1 - q^{4}}\right)\right\}$$

The left hand side (LHS) factorial is expanded into q form as well,

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q^{\sim} = \frac{[n]^{\sim}[n-1]^{\sim}}{[2]^{\sim}} = \frac{(q^n - q^{-n})(q^{n-1} - q^{-n+1})}{(q^1 - q^{-1})(q^2 - q^{-2})}$$

$$= \frac{(q^{2N+m} - q^{-2N-m})(q^{2N+m-1} - q^{-2N-m+1})}{(q^1 - q^{-1})(q^2 - q^{-2})}$$

The difference between the LHS and the RHS is,

$$\frac{q^1 + q^{-1} - q^{2m-5} - q^{-2m+5}}{(q^1 - q^{-1})(q^2 - q^{-2})}$$

which is zero for both m = 2 & 3. Thus, the LHS and RHS are equivalent.

Chapter 3

[n choose 3]˜ Equations

The form for k = 3 is not given as easily. A Mathematica code was written to output the list of q-bracketed numbers for n choose 3 (Appendix A). This list was then manually analyzed until a pattern was found. The key is the mixed spacing of 0, 2, 4 or 6 between the bracketed numbers. It then became apparent that there were overlapping patterns. These overlapping patterns can be expressed as a double summation.

Initial Summation Equation

Initially n choose 3 was found to be numerically equivalent to a set of three equations, each for a subset of n. The equations contain additional complication as certain terms are only applicable beyond certain values of n. The notation below shows each term and the range of n values where it is valid.

For n = 4, 6, 8, 10 ...

$$\begin{bmatrix} n \\ k=3 \end{bmatrix}_q^{\sim} = [\mathrm{k}(\mathrm{n}-\mathrm{k})+1]_q^{\sim} + \begin{cases} \displaystyle\sum_{m=1}^{\left(\frac{3n}{2}-7\right)} [\mathrm{k}(\mathrm{n}-\mathrm{k})-1-2m]_q^{\sim}, & n \geq 6 \\ \displaystyle + \sum_{i=1}^{\left(\frac{n-8}{2}\right)} \sum_{j=1}^{i} [2i+4+4j]_q^{\sim}, & n \geq 10 \end{cases}$$

For n = 3, 7, 11, 15 …

$$\begin{bmatrix} n \\ k=3 \end{bmatrix}_q^{\sim} = [\mathrm{k(n-k)+1}]_q^{\sim} + \begin{cases} [1]_q^{\sim} + \displaystyle\sum_{m=1}^{\left(\frac{3(n-3)}{2}-3\right)} [\mathrm{k(n-k)} - 1 - 2m]_q^{\sim}, & n \geq 7 \\ + \displaystyle\sum_{i=0}^{\left(\frac{n-11}{4}\right)} \sum_{j=0}^{2i+1} [4i + 9 + 4j]_q^{\sim}, & n \geq 11 \\ + \displaystyle\sum_{i=0}^{\left(\frac{n-15}{4}\right)} \sum_{j=0}^{2i+1} [4i + 15 + 4j]_q^{\sim}, & n \geq 15 \end{cases}$$

For n=5, 9, 13, 17 …

$$\begin{bmatrix} n \\ k=3 \end{bmatrix}_q^{\sim} = [\mathrm{k(n-k)+1}]_q^{\sim} + [3]_q^{\sim} + \begin{cases} \displaystyle\sum_{m=1}^{\left(\frac{3(n-3)}{2}-4\right)} [\mathrm{k(n-k)} - 1 - 2m]_q^{\sim}, & n \geq 5 \\ + \displaystyle\sum_{i=0}^{\left(\frac{n-9}{4}\right)} \sum_{j=0}^{2i} [4i + 7 + 4j]_q^{\sim}, & n \geq 9 \\ + \displaystyle\sum_{i=0}^{\left(\frac{n-13}{4}\right)} \sum_{j=0}^{2i} [4i + 13 + 4j]_q^{\sim}, & n \geq 13 \end{cases}$$

These equations have been symbolically verified using a computer script to a value of n greater than 100. The code used to accomplish this task is available in Appendix B. This confirmation shows that the equations are very likely correct as no new trends are expected to develop in the system past a large value of n.

A More Elegant Solution

The equations above appear to work, but are cumbersome. After further analysis, a new single equation was developed. This equation contains a single double summation with floor functions at the upper limits. These floor functions significantly reduce the complexity of the formulation; however, they will make the proof of this equation more difficult.

$$\begin{bmatrix} n \\ k=3 \end{bmatrix}_q^{\sim} = \sum_{j=0}^{f} \sum_{i=0}^{g} [\mathrm{k(n-k)} + 1 - 4i - 6j]_q^{\sim}$$

where,

$$f = \left\lfloor \frac{n-k}{k} \right\rfloor$$

$$g = \left\lfloor \frac{n-k-3j}{2} \right\rfloor + \left\lfloor \frac{n-k-3j}{4} \right\rfloor$$

This new function has also been verified using a symbolic computer script to a high value of n, again this showed that these equations are very likely correct (code is available in Appendix C). In order to show that this equation is true for all n ≥ 3 (n is in the set of integers) a mathematical proof is required.

Chapter 4

Proof of Simplified [n choose 3]˜ Equation

Goal and Plan

The elegant double summation equation was shown with symbolic numerical techniques to equal the factorial. A mathematical proof is required to shown that this equation holds for all n ≥ 3 (n is in the set of integers).

$$\begin{bmatrix} n \\ k=3 \end{bmatrix}_q^{\sim} = \frac{[n]^{\sim}[n-1]^{\sim}[n-2]^{\sim}}{[3]^{\sim}[2]^{\sim}} \overset{?}{=} \sum_{j=0}^{f(n)} \sum_{i=0}^{g(n,j)} [3(\mathrm{n}-3)+1-4i-6j]_q^{\sim}$$

This equation is converted into q form so that the LHS and RHS can be shown to be equal.

$$\frac{\left(q^n - \frac{1}{q^n}\right)\left(-\frac{1}{q^{n-1}} + q^{n-1}\right)\left(-\frac{1}{q^{n-2}} + q^{n-2}\right)}{\left(-\frac{1}{q} + q\right)\left(-\frac{1}{q^2} + q^2\right)\left(-\frac{1}{q^3} + q^3\right)} \overset{?}{=}$$

$$\sum_{j=0}^{f(n)} \sum_{i=0}^{g(n,j)} \left(\frac{q^{(3(\mathrm{n}-3)+1-4i-6j)} - q^{-(3(\mathrm{n}-3)+1-4i-6j)}}{q - q^{-1}}\right)$$

The limits on the summations pose the greatest difficulty. They contain floor functions which do not lend themselves to easy manipulation. The next section will focus on the reduction of these summations and the removal of the floor functions.

Summation Reduction

The first summation is only a function of i. As a result is can be decomposed into an enclosed form. This reduces the equation to a single summation. Beginning with the inside summation, the equation is put into q form. All variables that are not a function of i can be pulled out of the inside summation. The terms n, g(n,j), and j are constant, only i is varied.

$$\sum_{i=0}^{g(n,j)} \frac{q^{3(\mathrm{n}-3)+1-4i-6j} - q^{-(3(\mathrm{n}-3)+1-4i-6j)}}{q - q^{-1}}$$

The terms independent of i are pulled from the summations.

$$\left(\frac{q^{3(\mathrm{n}-3)+1-6j}}{q - q^{-1}}\right) \sum_{i=0}^{g(n,j)} q^{-4i} - \left(\frac{q^{-(3(\mathrm{n}-3)+1-6j)}}{q - q^{-1}}\right) \sum_{i=0}^{g(n,j)} q^{4i}$$

Using the geometric series the summations are reduced. The general form of the geometric series is given as,

$$\sum_{k=0}^{n} ar^{bk} = a\frac{1 - r^{b(n+1)}}{1 - r^{b}}$$

This is applied to the two summations above.

$$\sum_{i=0}^{g(n,j)} q^{-4i} = \left(\frac{1 - q^{-4(g(n,j)+1)}}{1 - q^{-4}}\right)$$

$$\sum_{i=0}^{g(n,j)} q^{4i} = \left(\frac{1 - q^{4(g(n,j)+1)}}{1 - q^{4}}\right)$$

In order to insure that the summation upper bound is greater than zero it is shown that the function g is always greater than or equal to zero as,

$$g_{min} = \left\lfloor\frac{n - 3 - 3j_{max}}{2}\right\rfloor + \left\lfloor\frac{n - 3 - 3\,j_{max}}{4}\right\rfloor$$

$$j_{max} = f(n) = \left\lfloor\frac{n - 3}{3}\right\rfloor$$

$$g_{min} = \left\lfloor \left(\frac{3}{2}\right)\left(\frac{n-3}{3} - \left\lfloor \frac{n-3}{3} \right\rfloor\right) \right\rfloor + \left\lfloor \left(\frac{3}{4}\right)\left(\frac{n-3}{3} - \left\lfloor \frac{n-3}{3} \right\rfloor\right) \right\rfloor$$

and because $n \geq 3$,

$$\frac{n-3}{3} \geq \left\lfloor \frac{n-3}{3} \right\rfloor$$

Therefore g is always zero or greater. Now the inside summation is only in terms of q, g, n and j. The variable, i, has been removed. The total equation is now in the form,

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^{\sim} = \sum_{j=0}^{f} \left\{ \left(\frac{q^{3(\mathrm{n}-3)+1-6j}}{q-q^{-1}}\right)\left(\frac{1-q^{-4(g+1)}}{1-q^{-4}}\right) - \left(\frac{q^{-(3(\mathrm{n}-3)+1-6j)}}{q-q^{-1}}\right)\left(\frac{1-q^{4(g+1)}}{1-q^{4}}\right) \right\}$$

where,

$$f = \left\lfloor \frac{n-3}{3} \right\rfloor$$

$$g = \left\lfloor \frac{n-3-3j}{2} \right\rfloor + \left\lfloor \frac{n-3-3j}{4} \right\rfloor$$

Because this summation contains floor functions g and f and g is a function of j, the summation cannot be reduced immediately using geometric series. Therefore, in order to decompose the floor functions to allow for a proof n and j must be broken up into sub groups.

The functions f and g have periods of three and four respectively. Thus, the common period of n which will decompose both terms is 12. The variable j is only in the function g, thus j only needs to be decomposed by a period of 4.

$$n = 12N + m, \qquad m = 3,4,5, \dots 14$$

$$j = 4H + p \qquad p = 0,1,2,3$$

Splitting j into a function of H and p splits the reduced single summation back into a double summation.

$$\sum_{j=0}^{f} a[j] = \sum_{p=0}^{3} \sum_{H=0}^{H_{max}} a[H,p]$$

$$f = \left\lfloor \frac{n-3}{3} \right\rfloor = \left\lfloor \frac{12N + m - 3}{3} \right\rfloor = 4N + f_2(m)$$

$$f_2 = \left\lfloor \frac{m-3}{3} \right\rfloor$$

The limit on H must be made to match the original limit, f. Table 2 below shows the generation of j with the expansion 4H + p. The limit of j is $4N + f_2$, thus depending upon the value of p, H is limited at N or N - 1. This is provided by the function,

$$H_{max} = N - L(p,m)$$

$$L(p,m) = \begin{cases} 0, p \le f_2 \\ 1, p > f_2 \end{cases}$$

This piecewise function will create a problem which will prevent a general solution for all m and N. However, because m and p are finite set of integers there are a limited number of cases to be individually solved in the end.

Table 2. Values of j expanded with 4H + p

	p = 0	**p = 1**	**p = 2**	**p = 3**
H = 1	0	1	2	3
H = 2	4	5	6	7
…				
H = N - 1	4N - 4	4N - 3	4N - 2	4N - 1
H = N	4N	4N + 1	4N + 2	4N + 3

Finally, the function g is rewritten with the expanded variables as well.

$$g = \left\lfloor \frac{n-3-3j}{2} \right\rfloor + \left\lfloor \frac{n-3-3j}{4} \right\rfloor$$

$$g = \left\lfloor \frac{12N+m-3-3(4H+p)}{2} \right\rfloor + \left\lfloor \frac{12N+m-3-3(4H+p)}{4} \right\rfloor$$

$$g = 9N - 9H + l(p,m)$$

where,

$$l(p,m) = \left\lfloor \frac{m-3-3p}{2} \right\rfloor + \left\lfloor \frac{m-3-3p}{4} \right\rfloor$$

A new variable l(p,m) is presented. Its values are shown in Table 3.

The total equation was in the form,

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^{\sim} = \left(\frac{1}{q-q^{-1}}\right) \sum_{p=0}^{3} \sum_{H=0}^{N-L(p,m)} \left\{ q^{3(\mathrm{n}-3)+1-6j} \left(\frac{1-q^{-4(g+1)}}{1-q^{-4}}\right) - q^{-(3(\mathrm{n}-3)+1-6j)} \left(\frac{1-q^{4(g+1)}}{1-q^{4}}\right) \right\}$$

Table 3. l(p,m)

m	3	4	5	6	7	8	9	10	11	12	13	14
p = 0	0	0	1	1	3	3	4	4	6	6	7	7
p = 1	-3	-2	-2	0	0	1	1	3	3	4	4	6
p = 2	-5	-5	-3	-3	-2	-2	0	0	1	1	3	3
p = 3	-8	-6	-6	-5	-5	-3	-3	-2	-2	0	0	1

The expansions on n and j are applied yielding,

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^{\sim} = \left(\frac{1}{q - q^{-1}}\right) \sum_{p=0}^{3} \sum_{H=0}^{N-L(p,m)} \left\{ q^{(36\mathrm{N}+3\mathrm{m}-8-24\mathrm{H}-6\mathrm{p})} \left(\frac{1 - q^{-4(g+1)}}{1 - q^{-4}}\right) - q^{-(36\mathrm{N}+3\mathrm{m}-8-24\mathrm{H}-6\mathrm{p})} \left(\frac{1 - q^{4(g+1)}}{1 - q^{4}}\right) \right\}$$

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^{\sim} = \left(\frac{1}{q - q^{-1}}\right) \sum_{p=0}^{3} \sum_{H=0}^{N-L(p,m)} \left\{ \left(\frac{q^{(36\mathrm{N}+3\mathrm{m}-8-6\mathrm{p})}}{1 - q^{-4}}\right) q^{-24H} \left(1 - q^{-4(g+1)}\right) - \left(\frac{q^{-(36\mathrm{N}+3\mathrm{m}-8-6\mathrm{p})}}{1 - q^{4}}\right) q^{24H} \left(1 - q^{4(g+1)}\right) \right\}$$

The two key summations within this expanded form are,

$$\sum_{H=0}^{N-L(p,m)} q^{-24H} \left(1 - q^{-4(g+1)}\right)$$

and,

$$\sum_{H=0}^{N-L(p,m)} q^{24H} \left(1 - q^{4(g+1)}\right)$$

These equations are reduced using the geometric series as shown before.

$$\sum_{H=0}^{N-L(p,m)} q^{-24H} \left(1 - q^{-4(g+1)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{-24H} \left(1 - q^{-4(9N-9H+l(p,m)+1)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} \left(q^{-24H} - q^{12H} q^{(-36N-4l(p,m)-4)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{-24H} = \left(\frac{1-q^{-24(N-L(p,m)+1)}}{1-q^{-24}}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{12H} = \left(\frac{1-q^{12(N-L(p,m)+1)}}{1-q^{12}}\right)$$

The second summation is similar to the first.

$$\sum_{H=0}^{N-L(p,m)} q^{24H}\left(1-q^{4(g+1)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{24H}\left(1-q^{4(9N-9H+l(p,m)+1)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} \left(q^{24H}-q^{-12H}q^{(36N+4l(p,m)+4)}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{24H} = \left(\frac{1-q^{24(N-L(p,m)+1)}}{1-q^{24}}\right)$$

$$\sum_{H=0}^{N-L(p,m)} q^{-12H} = \left(\frac{1-q^{-12(N-L(p,m)+1)}}{1-q^{-12}}\right)$$

Putting it all together,

$$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^{\sim} = \left(\frac{1}{q-q^{-1}}\right)\sum_{p=0}^{3}\left\{\left(\frac{q^{(36\mathrm{N}+3\mathrm{m}-8-6\mathrm{p})}}{1-q^{-4}}\right)\left[\left(\frac{1-q^{-24(N-L(p,m)+1)}}{1-q^{-24}}\right)\right.\right.$$

$$\left. - q^{(-36N-4l(p,m)-4)}\left(\frac{1-q^{12(N-L(p,m)+1)}}{1-q^{12}}\right)\right]$$

$$-\left(\frac{q^{-(36\mathrm{N}+3\mathrm{m}-8-6\mathrm{p})}}{1-q^{4}}\right)\left[\left(\frac{1-q^{24(N-L(p,m)+1)}}{1-q^{24}}\right)\right.$$

$$\left.\left. - q^{(36N+4l(p,m)+4)}\left(\frac{1-q^{-12(N-L(p,m)+1)}}{1-q^{-12}}\right)\right]\right\}$$

This reduced equation is only a function of p, m, and N. At this point, because the functions L(p,m) and l(p,m) cannot be reduced in order to isolate the summation variable p, this summation will have be manually expanded. Fortunately there are only 4 terms in this summation, a product of the two previous summation reductions. Applying this expansion to the case where m = 3 yields,

For m = 3 and p = 0: $f_2 = 0$, $L = 0$, $l = 0$,

$$\left\{\left(\frac{q^{(36\mathrm{N}+1)}}{1-q^{-4}}\right)\left[\left(\frac{1-q^{-24(N+1)}}{1-q^{-24}}\right) - q^{(-36N-4)}\left(\frac{1-q^{12(N+1)}}{1-q^{12}}\right)\right]\right.$$

$$\left. -\left(\frac{q^{-(36\mathrm{N}+1)}}{1-q^{4}}\right)\left[\left(\frac{1-q^{24(N+1)}}{1-q^{24}}\right) - q^{(36N+4)}\left(\frac{1-q^{-12(N+1)}}{1-q^{-12}}\right)\right]\right\}$$

For m = 3 and p = 1: f_2 = 0, L = 1, l = -3,

$$\left\{\left(\frac{q^{(36N-5)}}{1-q^{-4}}\right)\left[\left(\frac{1-q^{-24(N)}}{1-q^{-24}}\right)-q^{(-36N+8)}\left(\frac{1-q^{12(N)}}{1-q^{12}}\right)\right]\right.$$
$$\left.-\left(\frac{q^{-(36N-5)}}{1-q^{4}}\right)\left[\left(\frac{1-q^{24(N)}}{1-q^{24}}\right)-q^{(36N-8)}\left(\frac{1-q^{-12(N)}}{1-q^{-12}}\right)\right]\right\}$$

For m = 3 and p = 2: f_2 = 0, L = 1, l = -5,

$$\left\{\left(\frac{q^{(36N-11)}}{1-q^{-4}}\right)\left[\left(\frac{1-q^{-24(N)}}{1-q^{-24}}\right)-q^{(-36N+16)}\left(\frac{1-q^{12(N)}}{1-q^{12}}\right)\right]\right.$$
$$\left.-\left(\frac{q^{-(36N-11)}}{1-q^{4}}\right)\left[\left(\frac{1-q^{24(N)}}{1-q^{24}}\right)-q^{(36N-16)}\left(\frac{1-q^{-12(N)}}{1-q^{-12}}\right)\right]\right\}$$

For m = 3 and p = 3: f_2 = 0, L = 1, l = -8,

$$\left\{\left(\frac{q^{(36N-17)}}{1-q^{-4}}\right)\left[\left(\frac{1-q^{-24(N)}}{1-q^{-24}}\right)-q^{(-36N+28)}\left(\frac{1-q^{12(N)}}{1-q^{12}}\right)\right]\right.$$
$$\left.-\left(\frac{q^{-(36N-17)}}{1-q^{4}}\right)\left[\left(\frac{1-q^{24(N)}}{1-q^{24}}\right)-q^{(36N-28)}\left(\frac{1-q^{-12(N)}}{1-q^{-12}}\right)\right]\right\}$$

These four terms are summed and then multiplied by $(q - q^{-1})^{-1}$. With the aid of a symbolic manipulation code (available in Appendix D) this is shown to equal, (for m = 3),

$$\frac{(q^{12N+3}-\frac{1}{q^{12N+3}})(-\frac{1}{q^{12N+2}}+q^{12N+2})(-\frac{1}{q^{12N+1}}+q^{12N+1})}{(-\frac{1}{q}+q)(-\frac{1}{q^2}+q^2)(-\frac{1}{q^3}+q^3)}$$

Which proves that the summation equation is equal to the second quantum binomial coefficient for m = 3. The evaluation of the p-summation is repeated (Appendix E) for each value of m and is shown to be valid in all cases. Thus for all N ≥ 0 and m = 3, 4, …14 the summation equation is equal to the second quantum binomial coefficient.

Chapter 5

Classical Limits and Future Work

Classical Limits

The proof above is valid for $0 < q < 1$, thus as $q \to 1^-$ the solution converges to the solution for the classical binomial coefficient. In the classical limit there are no longer discrete bracketed values, instead the integers add, forming the final value of the binomial coefficient. Therefore,

$$\binom{n}{k=2} = \sum_{i=0}^{f} (\mathrm{k}(\mathrm{n}-\mathrm{k}) + 1 - 4i)$$

where,

$$f = \left\lfloor \frac{n-2}{2} \right\rfloor$$

and,

$$\binom{n}{k=3} = \sum_{j=0}^{f} \sum_{i=0}^{g} (\mathrm{k}(\mathrm{n}-\mathrm{k}) + 1 - 4i - 6j)$$

where,

$$f = \left\lfloor \frac{n-k}{k} \right\rfloor$$

$$g = \left\lfloor \frac{n-k-3j}{2} \right\rfloor + \left\lfloor \frac{n-k-3j}{4} \right\rfloor$$

[n choose 4]~ and Beyond

The form given with k = 2 and 3 shows a pattern emerging. Following this pattern yields something of the form for k = 4,

$$\begin{bmatrix} n \\ k=4 \end{bmatrix}_q^{\sim} = \sum_{h=0}^{f} \sum_{j=0}^{g} \sum_{i=0}^{e} [\mathrm{k(n-k)} + 1 - 4i - 6j - 8h]_q^{\sim}$$

If the previous patterns continue it is supposed that,

$$f = \left\lfloor \frac{n-k}{k} \right\rfloor$$

$$g \overset{?}{=} \left\lfloor \frac{n-k-kh}{k-1} \right\rfloor + \left\lfloor \frac{n-k-kh}{k+1} \right\rfloor$$

$$e \overset{?}{=} \left\lfloor \frac{n-k-C_1h-C_2j}{k-2} \right\rfloor + \left\lfloor \frac{n-k-C_1h-C_2j}{k} \right\rfloor + \left\lfloor \frac{n-k-C_1h-C_2j}{k+2} \right\rfloor$$

where f and g follow the previous form and e presents a new challenge. The work in the previous chapters shows that further analysis may uncover a solution which is general for all k which will have k-1 summations, with some structure defining the bounds. The summation bounds were determined for k = 2 and k = 3. Uncovering the k = 4 solution would assist in uncovering the underlying structure in the floor limits for all k.

Finally, similar analysis as shown in this thesis is likely possible for other quantum forms of the binomial coefficient and will be explored.

Chapter 6

Conclusion

A unique form of the quantum binomial coefficient [n choose k]˜ for k = 2 and 3 has been shown. The equations both show the discrete nature of the quantum form as the binomial coefficient is partitioned into specific quantum integers. The proof of these equations has been shown as well. The equations show that a general form of the quantum binomial coefficient with k summations appears to be feasible. This will be investigated in future work.

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Appendix

Appendix A

Generation of Quantum Integers for the Second Q-Binomial Coefficient (Mathematica Code)

```
Clear[k,n,nn,Nmax,i,j,h,k,e,q,f,g,e2,e1,p,count,countu,QuantsFull];
Quant[n_,q_]:=(1-q^n)/(1-q);
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

n=9;
k=3;

Array[QuantsFull,Binomial[n,k]];
count=0;
countu=0;

e
=Normal[Series[Expand[Simplify[(Product[Quant2[i,q],{i,2,n}])/((Product
[Quant2[j,q],{j,2,k}])*(Product[Quant2[h,q],{h,2,n-k}]))]],{q,0,(n-
k)*k+10}]]

f=Exponent[e,q];

If[f==0,
  QuantsFull[1]=1;
  count=1;
  countu=1;
  ];
e1=e;
p=1;

While[ f>0,
  count=count+1;

  QuantsFull[p]=f+1;

  g=e1-Quant2[(f+1),q];
  e2 =Normal[Series[Expand[Simplify[g]],{q,0,(n-k)*k+10}]];

  If[f==Exponent[e2,q],
   countu=countu;,
   countu=countu+1;
   ];

  f=Exponent[e2,q];
  p=p+1;

  If[f==0,
   ones=e2;
   For[pp=1,pp<=(ones-1),pp++,
    QuantsFull[p]=1;
    e2=e2-1;
    p=p+1;
    count=count+1;
```

```
    countu=countu+1;
    ];

   QuantsFull[p]=1;
   count=count+1;
   countu=countu+1;
   ];

  e1=e2;
  ];
Print[Style["k=",24,Blue],Style[k,18,Blue],Style["
n=",18,Red],Style[n,18,Red]];
Print[Table[QuantsFull[l],{l,count}]];
```

Example Output (for the input conditions shown in the code above):

$$8+\frac{1}{q^{18}}+\frac{1}{q^{16}}+\frac{2}{q^{14}}+\frac{3}{q^{12}}+\frac{4}{q^{10}}+\frac{5}{q^{8}}+\frac{7}{q^{6}}+\frac{7}{q^{4}}+\frac{8}{q^{2}}+8q^2+7q^4+7q^6+5q^8+4q^{10}+3q^{12}+2q^{14}+q^{16}+q^{18}$$

```
k = 3   n = 9
{19,15,13,11,9,7,7,3}
```

Appendix B

For n is Even (Mathematica Code)

```
Clear[k,n,Nmax,i,j,h,e,q,f,g,e2,e1,p,count,countu,QuantsFull,m,s,r];

Quant[n_,q_]:=(1-q^n)/(1-q);
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

(*n is even*)
n=16;
k=3;

Array[QuantsFull,Binomial[n,k]];

Print[Style["k=",24,Blue],Style[k,18,Blue]]

count=0;
countu=0;
Print[Style["n=",18,Red],Style[n,18,Red]];

e
=Normal[Series[Expand[Simplify[(Product[Quant2[i,q],{i,2,n}])/((Product
[Quant2[j,q],{j,2,k}])*(Product[Quant2[h,q],{h,2,n-k}]))]],{q,0,(n-
k)*k+10}]]

Print[e]
f=Exponent[e,q];

If[f==0,
  QuantsFull[1]=1;
  count=1;
  countu=1;
  ];

e1=e;
p=1;

While[ f>0,
  count=count+1;

  QuantsFull[p]=f+1;

  g=e1-Quant2[(f+1),q];
  e2 =Normal[Series[Expand[Simplify[g]],{q,0,1001}]];

  If[f==Exponent[e2,q],
   countu=countu;,
   countu=countu+1;
   ];

  f=Exponent[e2,q];
  p=p+1;

  If[f==0,
```

```
   QuantsFull[p]=1;

   count=count+(1*Normal[e2]);
   countu=countu+1;
   ];

  e1=e2;
  ];

Print[Table[QuantsFull[l],{l,count}]];

(*Print[Table[QuantsFull[l]-QuantsFull[l+1],{l,count-1}]];*)

PRE=Table[0,{j,1,(((n-1)^2)/8-1/8)}];
spot=1;
PRE[[spot]]=(n-k)*k+1;
spot=spot+1;
(*Print[PRE]*)

If[n>=6,
 For[m=1,m<=((3*n)/2-7),m++,
   PRE[[spot]]=k*(n-k)-1-2*m;
   spot=spot+1;];]
(*Print[PRE]*)

If[n>=10,
 For[s=1,s<=(((n-8)/2)),s++,
   For[r=1,r<=s,r++,
    PRE[[spot]]=4+2*s+4*r ;
    spot=spot+1;
    ]
   ];]

PRE2=Sort[PRE,Greater];
Print[Style[PRE2,Green]];
DIFFS=Table[5,{o,1,((n-1)^2/8-1/8)}];
For[L=1,L<=((n-1)^2/8-1/8),L++,
 DIFFS[[L]]=QuantsFull[L]-PRE2[[L]];]
(*Difference table should be all zeros!!!*)
Print[Style[DIFFS,Red]]
```

Output (with the input conditions shown in the code above):

k = 3

n = 16

$$\frac{1}{q^{39}}+\frac{1}{q^{37}}+\frac{2}{q^{35}}+\frac{3}{q^{33}}+\frac{4}{q^{31}}+\frac{5}{q^{29}}+\frac{7}{q^{27}}+\frac{8}{q^{25}}+\frac{10}{q^{23}}+\frac{12}{q^{21}}+\frac{14}{q^{19}}+\frac{16}{q^{17}}+\frac{19}{q^{15}}+\frac{21}{q^{13}}+\frac{23}{q^{11}}+\frac{25}{q^{9}}+\frac{26}{q^{7}}+\frac{27}{q^{5}}+\frac{28}{q^{3}}+\frac{28}{q}+28q+28q^3+27q^5+26q^7+25q^9+23q^{11}+21q^{13}+19q^{15}+16q^{17}+14q^{19}+12q^{21}+10q^{23}+8q^{25}+7q^{27}+5q^{29}+4q^{31}+3q^{33}+2q^{35}+q^{37}+q^{39}$$

$$\frac{1}{q^{39}}+\frac{1}{q^{37}}+\frac{2}{q^{35}}+\frac{3}{q^{33}}+\frac{4}{q^{31}}+\frac{5}{q^{29}}+\frac{7}{q^{27}}+\frac{8}{q^{25}}+\frac{10}{q^{23}}+\frac{12}{q^{21}}+\frac{14}{q^{19}}+\frac{16}{q^{17}}+\frac{19}{q^{15}}+\frac{21}{q^{13}}+\frac{23}{q^{11}}+\frac{25}{q^{9}}+\frac{26}{q^{7}}+\frac{27}{q^{5}}+\frac{28}{q^{3}}+\frac{28}{q}+28q+28q^3+27q^5+26q^7+25q^9+23q^{11}+21q^{13}+19q^{15}+16q^{17}+14q^{19}+12q^{21}+10q^{23}+8q^{25}+7q^{27}+5q^{29}+4q^{31}+3q^{33}+2q^{35}+q^{37}+q^{39}$$

```
{40,36,34,32,30,28,28,26,24,24,22,22,20,20,18,18,16,16,16,14,14,12,12,1
0,10,8,6,4}
{40,36,34,32,30,28,28,26,24,24,22,22,20,20,18,18,16,16,16,14,14,12,12,1
0,10,8,6,4}
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
```

For n =4*nn+3 (Mathematica Code)

```
Clear[k,n,Nmax,i,j,h,e,q,f,g,e2,e1,p,count,countu,QuantsFull,mi,mj,PRE,
spot,s,r,m];

Quant[n_,q_]:=(1-q^n)/(1-q);
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

nn=2;
n=4*nn+3;
k=3;

Array[QuantsFull,Binomial[n,k]];

Print[Style["k=",24,Blue],Style[k,18,Blue]]

count=0;
countu=0;
Print[Style["n=",18,Red],Style[n,18,Red]];
e
=Normal[Series[Expand[Simplify[(Product[Quant2[i,q],{i,2,n}])/((Product
[Quant2[j,q],{j,2,k}])*(Product[Quant2[h,q],{h,2,n-k}]))]],{q,0,(n-
k)*k+10}]]

Print[e]
f=Exponent[e,q];

If[f==0,
  QuantsFull[1]=1;
  count=1;
  countu=1;
  ];

e1=e;
p=1;

While[ f>0,
  count=count+1;

  QuantsFull[p]=f+1;

  g=e1-Quant2[(f+1),q];
  e2 =Normal[Series[Expand[Simplify[g]],{q,0,201}]];

  If[f==Exponent[e2,q],
   countu=countu;,
   countu=countu+1;
   ];

  f=Exponent[e2,q];
  p=p+1;

  If[f==0,
   QuantsFull[p]=1;
```

```
   count=count+(1*Normal[e2]);
   countu=countu+1;
   ];

  e1=e2;
  ];

Print[Table[QuantsFull[l],{l,count}]];

PRE=Table[0,{j,1,(((n-1)^2)/8+1/2)}];
spot=1;
PRE[[spot]]=(n-k)*k+1;
spot=spot+1;

If[n>=7,
 PRE[[spot]]=1;
 spot=spot+1;
 For[m=1,m<=((k*(n-k)-6)/2),m++,
  PRE[[spot]]=k*(n-k)-1-2*m;
  spot=spot+1;];]

If[n>=11,
 For[s=0,s<=((n-11)/4),s++,
   For[r=0,r<=(2*s+1),r++,
    PRE[[spot]]=9+4*s+4*r ;
    spot=spot+1;
    ]
   ];]

If[n>=15,
 For[s=0,s<=((n-15)/4),s++,
   For[r=0,r<=(2*s+1),r++,
    PRE[[spot]]=15+4*s+4*r ;
    spot=spot+1;
    ]
   ];]

PRE2=Sort[PRE,Greater];
Print[Style[PRE2,Green]];
DIFFS=Table[5,{o,1,((n-1)^2/8+1/2)}];
For[L=1,L<=((n-1)^2/8+1/2),L++,
 DIFFS[[L]]=QuantsFull[L]-PRE2[[L]];]
(*Difference table should be all zeros!!!*)
Print[Style[DIFFS,Red]]
```

Output (with the input conditions shown in the code above):

k= 3 n= 11

$$13+\frac{1}{q^{24}}+\frac{1}{q^{22}}+\frac{2}{q^{20}}+\frac{3}{q^{18}}+\frac{4}{q^{16}}+\frac{5}{q^{14}}+\frac{7}{q^{12}}+\frac{8}{q^{10}}+\frac{10}{q^{8}}+\frac{11}{q^{6}}+\frac{12}{q^{4}}+\frac{12}{q^{2}}+12q^2+12q^4+11q^6+10q^8+8q^{10}+7q^{12}+5q^{14}+4q^{16}+3q^{18}+2q^{20}+q^{22}+q^{24}$$

{25,21,19,17,15,13,13,11,9,9,7,5,1}

{25,21,19,17,15,13,13,11,9,9,7,5,1}

{0,0,0,0,0,0,0,0,0,0,0,0,0}

For n =4*nn+3 (Mathematica Code)

```
Clear[k,n,Nmax,i,j,h,e,q,f,g,e2,e1,p,count,countu,QuantsFull,mi,mj,PRE,
spot,s,r,m];

Quant[n_,q_]:=(1-q^n)/(1-q);
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

nn=2;
n=4*nn+5;
k=3;

Array[QuantsFull,Binomial[n,k]];

Print[Style["k=",24,Blue],Style[k,18,Blue]]

count=0;
countu=0;
Print[Style["n=",18,Red],Style[n,18,Red]];
e
=Normal[Series[Expand[Simplify[(Product[Quant2[i,q],{i,2,n}])/((Product
[Quant2[j,q],{j,2,k}])*(Product[Quant2[h,q],{h,2,n-k}]))]],{q,0,(n-
k)*k+10}]]
Print[e]
f=Exponent[e,q];

If[f==0,
  QuantsFull[1]=1;
  count=1;
  countu=1;
  ];

e1=e;
p=1;

While[ f>0,
  count=count+1;

  QuantsFull[p]=f+1;

  g=e1-Quant2[(f+1),q];
  e2 =Normal[Series[Expand[Simplify[g]],{q,0,301}]];

  If[f==Exponent[e2,q],
   countu=countu;,
   countu=countu+1;
   ];

  f=Exponent[e2,q];
  p=p+1;

  If[f==0,
   QuantsFull[p]=1;

   count=count+(1*Normal[e2]);
```

```
   countu=countu+1;
   ];

  e1=e2;
  ];

Print[Table[QuantsFull[l],{l,count}]];

PRE=Table[0,{j,1,(((n-1)^2)/8)}];
spot=1;
PRE[[spot]]=(n-k)*k+1;
spot=spot+1;
PRE[[spot]]=3;
spot=spot+1;

If[n≥5,
 For[m=1,m≤((k*(n-k)-8)/2),m++,
   PRE[[spot]]=k*(n-k)-1-2*m;
   spot=spot+1;];]

If[n≥9,
 For[s=0,s≤((n-9)/4),s++,
   For[r=0,r≤(2*s),r++,
    PRE[[spot]]=7+4*s+4*r ;
    spot=spot+1;
    ]
   ];]

If[n≥13,
 For[s=0,s≤((n-13)/4),s++,
   For[r=0,r≤(2*s),r++,
    PRE[[spot]]=13+4*s+4*r ;
    spot=spot+1;
    ]
   ];]

PRE2=Sort[PRE,Greater];
Print[Style[PRE2,Green]];
DIFFS=Table[5,{o,1,((n-1)^2/8)}];
For[L=1,L<=((n-1)^2/8),L++,
 DIFFS[[L]]=QuantsFull[L]-PRE2[[L]];]
(*Difference table should be all zeros!!!*)
Print[Style[DIFFS,Red]]
```

Output (with the input conditions shown in the code above):

k= 3 n= 13

$$18 + \frac{1}{q^{30}} + \frac{1}{q^{28}} + \frac{2}{q^{26}} + \frac{3}{q^{24}} + \frac{4}{q^{22}} + \frac{5}{q^{20}} + \frac{7}{q^{18}} + \frac{8}{q^{16}} + \frac{10}{q^{14}} + \frac{12}{q^{12}} + \frac{14}{q^{10}} + \frac{15}{q^{8}} + \frac{17}{q^{6}} + \frac{17}{q^{4}} + \frac{18}{q^{2}} + 18 q^2 + 17 q^4 + 17 q^6 + 15 q^8 + 14 q^{10} + 12 q^{12} + 10 q^{14} + 8 q^{16} + 7 q^{18} + 5 q^{20} + 4 q^{22} + 3 q^{24} + 2 q^{26} + q^{28} + q^{30}$$

{31,27,25,23,21,19,19,17,15,15,13,13,11,11,9,7,7,3}
{31,27,25,23,21,19,19,17,15,15,13,13,11,11,9,7,7,3}
{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

Appendix C

Elegant [n choose 3]~ Numerical Verification (Mathematica Code)

```
Clear[k,n,nn,Nmax,i,j,h,k,e,q,f,g,e2,e1,p,count,countu,QuantsFull];

Quant[n_,q_]:=(1-q^n)/(1-q);
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

n=12*3+7;
k=3;

Array[QuantsFull,Binomial[n,k]];

count=0;
countu=0;

e
=Normal[Series[Expand[Simplify[(Product[Quant2[i,q],{i,2,n}])/((Product
[Quant2[j,q],{j,2,k}])*(Product[Quant2[h,q],{h,2,n-k}]))]],{q,0,(n-
k)*k+10}]]

f=Exponent[e,q];

If[f==0,
  QuantsFull[1]=1;
  count=1;
  countu=1;
  ];

e1=e;
p=1;

While[ f>0,
  count=count+1;

  QuantsFull[p]=f+1;

  g=e1-Quant2[(f+1),q];
  e2 =Normal[Series[Expand[Simplify[g]],{q,0,(n-k)*k+10}]];

  If[f==Exponent[e2,q],
   countu=countu;,
   countu=countu+1;
   ];

  f=Exponent[e2,q];
  p=p+1;

  If[f==0,
   ones=e2;
   For[pp=1,pp<=(ones-1),pp++,
    QuantsFull[p]=1;
```

```
     e2=e2-1;
     p=p+1;
     count=count+1;
     countu=countu+1;
     ];

    QuantsFull[p]=1;
    count=count+1;
    countu=countu+1;
    ];

   e1=e2;
   ];
Print[Style["k=",24,Blue],Style[k,18,Blue],Style["
n=",18,Red],Style[n,18,Red]];
Print[Table[QuantsFull[l],{l,count}]];

PRE=Table[0,{j,1,count}];
spot=1;

For[j=0,j<=Floor[(n-k)/k],j++,
  For[i=0,i<=(Floor[((n-k)-3*j)/2]+Floor[((n-k)-3*j)/4]),i++,
    PRE[[spot]]=(n-k)*k+1-4*i-6*j;
    spot=spot+1;
    ];];

PRE2=Sort[PRE,Greater];
Print[Style[PRE,Blue]];
DIFFS=Table[5,{o,1,count}];
For[L=1,L<=count,L++,
DIFFS[[L]]=QuantsFull[L]-PRE2[[L]];]
(*Difference table should be all zeros!!!*)
Print[Style[DIFFS,Red]]
```

Example Output (for the input conditions shown in the code above):

$$8+\frac{1}{q^{18}}+\frac{1}{q^{16}}+\frac{2}{q^{14}}+\frac{3}{q^{12}}+\frac{4}{q^{10}}+\frac{5}{q^{8}}+\frac{7}{q^{6}}+\frac{7}{q^{4}}+\frac{8}{q^{2}}+8q^2+7q^4+7q^6+5q^8+4q^{10}+3q^{12}+2q^{14}+q^{16}+q^{18}$$

```
k = 3   n = 9
{19,15,13,11,9,7,7,3}
{19,15,11,7,3,13,9,7}
{0,0,0,0,0,0,0,0}
```

Appendix D

m = 3 Reduction (Mathematica Code)

```
Clear[NN,m,q,n,f2,l,L,Quant2,e7,e6,th,e]
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

m=3;
k=3;

th=Simplify[(Quant2[12*NN+m,q]*Quant2[12*NN+m-1,q]*Quant2[12*NN+m-
2,q])/(Quant2[2,q]*Quant2[3,q])];

Simplify[th]

aa=(q^(36*NN+1)/(1-q^(-4))) (((1-q^(-24*(NN+1)))/(1-q^(-24)))-q^-
(36*NN+4) ((1-q^(12*(NN+1)))/(1-q^(12))))-(q^-(36*NN+1)/(1-q^(4)))
(((1-q^(24*(NN+1)))/(1-q^(24)))-q^(36*NN+4) ((1-q^(-12*(NN+1)))/(1-q^(-
12))))+(q^(36*NN-5)/(1-q^(-4))) (((1-q^(-24*(NN)))/(1-q^(-24)))-q^-
(36*NN-8) ((1-q^(12*(NN)))/(1-q^(12))))-(q^-(36*NN-5)/(1-q^(4))) (((1-
q^(24*(NN)))/(1-q^(24)))-q^(36*NN-8) ((1-q^(-12*(NN)))/(1-q^(-12))))+
   (q^(36*NN-11)/(1-q^(-4))) (((1-q^(-24*(NN)))/(1-q^(-24)))-q^-(36*NN-
16) ((1-q^(12*(NN)))/(1-q^(12))))-(q^-(36*NN-11)/(1-q^(4))) (((1-
q^(24*(NN)))/(1-q^(24)))-q^(36*NN-16) ((1-q^(-12*(NN)))/(1-q^(-
12))))+(q^(36*NN-17)/(1-q^(-4))) (((1-q^(-24*(NN)))/(1-q^(-24)))-q^-
(36*NN-28) ((1-q^(12*(NN)))/(1-q^(12))))-(q^-(36*NN-17)/(1-q^(4)))
(((1-q^(24*(NN)))/(1-q^(24)))-q^(36*NN-28) ((1-q^(-12*(NN)))/(1-q^(-
12))));
aa=aa/(q-q^-1);

Simplify[aa]

Simplify[aa-th]
```

Output:

$$\frac{q^{-36\,NN}\left(-1+q^{2+24\,NN}\right)\left(-1+q^{4+24\,NN}\right)\left(-1+q^{6+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)}$$

$$\frac{q^{-36\,NN}\left(-1+q^{2+24\,NN}+q^{4+24\,NN}+q^{6+24\,NN}-q^{6+48\,NN}-q^{8+48\,NN}-q^{10+48\,NN}+q^{12+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)}$$

0

Appendix E

m = 3 through 14 Reduction (Mathematica Code)

```
Clear[NN,m,q,n,f2,l,L,Quant2,e7,e6,LHS,e]
Quant2[n_,q_]:=(q^n-q^(-n))/(q-q^(-1));

For[m=3,m<=14,m++,
 k=3;

 LHS=Simplify[(Quant2[12*NN+m,q]*Quant2[12*NN+m-1,q]*Quant2[12*NN+m-
2,q])/(Quant2[2,q]*Quant2[3,q])];
 Simplify[LHS];

 f2=Floor[(m-3)/3];
 L[p_,m_]=If[p>f2,1,0];
 l[p_,m_]=Floor[(m-3-3*p)/2]+Floor[(m-3-3*p)/4];

 e6=0;
 For[p=0,p<=3,p++,
  L[p_,m_]=If[p>f2,1,0];
  l[p_,m_]=Floor[(m-3-3*p)/2]+Floor[(m-3-3*p)/4];
  e6=e6+((q^(36*NN+3*m-8-6*p)/(1-q^(-4))) (((1-q^(-24*(NN-
L[p,m]+1)))/(1-q^(-24)))-q^(-(36*NN+4*l[p,m]+4))*((1-q^(12*(NN-
L[p,m]+1)))/(1-q^(12))))-(q^(-(36*NN+3*m-8-6*p))/(1-q^(4))) (((1-
q^(24*(NN-L[p,m]+1)))/(1-q^(24)))-q^(36*NN+4*l[p,m]+4)*((1-q^(-(12*(NN-
L[p,m]+1))))/(1-q^(-12)))));
  ];

 e6=e6/(q-q^-1);
 Print[{Simplify[e6],Simplify[LHS],Simplify[e6-LHS]}]
  If[Simplify[e6-LHS]==0,Print["m = ",m,"   OK"],Print["m = ",m,"
ERROR"]];]
```

Output:

$$\left\{\frac{q^{-36\,NN}\left(-1+q^{2+24\,NN}+q^{4+24\,NN}+q^{6+24\,NN}-q^{6+48\,NN}-q^{8+48\,NN}-q^{10+48\,NN}+q^{12+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-36\,NN}\left(-1+q^{2+24\,NN}\right)\left(-1+q^{4+24\,NN}\right)\left(-1+q^{6+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 3 OK

$$\left\{\frac{q^{-3-36\,NN}\left(-1+q^{4+24\,NN}+q^{6+24\,NN}+q^{8+24\,NN}-q^{10+48\,NN}-q^{12+48\,NN}-q^{14+48\,NN}+q^{18+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-3-36\,NN}\left(-1+q^{4+24\,NN}\right)\left(-1+q^{6+24\,NN}\right)\left(-1+q^{8+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 4 OK

$$\left\{\frac{q^{-6-36\,NN}\left(-1+q^{6+24\,NN}+q^{8+24\,NN}+q^{10+24\,NN}-q^{14+48\,NN}-q^{16+48\,NN}-q^{18+48\,NN}+q^{24+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-6-36\,NN}\left(-1+q^{6+24\,NN}\right)\left(-1+q^{8+24\,NN}\right)\left(-1+q^{10+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 5 OK

$$\left\{\frac{q^{-9-36\,NN}\left(-1+q^{8+24\,NN}+q^{10+24\,NN}+q^{12+24\,NN}-q^{18+48\,NN}-q^{20+48\,NN}-q^{22+48\,NN}+q^{30+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-9-36\,NN}\left(-1+q^{8+24\,NN}\right)\left(-1+q^{10+24\,NN}\right)\left(-1+q^{12+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 6 OK

$$\left\{\frac{q^{-12-36\,NN}\left(-1+q^{10+24\,NN}+q^{12+24\,NN}+q^{14+24\,NN}-q^{22+48\,NN}-q^{24+48\,NN}-q^{26+48\,NN}+q^{36+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-12-36\,NN}\left(-1+q^{10+24\,NN}\right)\left(-1+q^{12+24\,NN}\right)\left(-1+q^{14+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 7 OK

$$\left\{\frac{q^{-15-36\,NN}\left(-1+q^{12+24\,NN}+q^{14+24\,NN}+q^{16+24\,NN}-q^{26+48\,NN}-q^{28+48\,NN}-q^{30+48\,NN}+q^{42+72\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ \frac{q^{-15-36\,NN}\left(-1+q^{12+24\,NN}\right)\left(-1+q^{14+24\,NN}\right)\left(-1+q^{16+24\,NN}\right)}{\left(-1+q^2\right)^3\left(1+2q^2+2q^4+q^6\right)},\ 0\right\}$$

m = 8 OK

$$\left\{\frac{q^{-18-36\,NN}\left(-1+q^{14+24\,NN}+q^{16+24\,NN}+q^{18+24\,NN}-q^{30+48\,NN}-q^{32+48\,NN}-q^{34+48\,NN}+q^{48+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-18-36\,NN}\left(-1+q^{14+24\,NN}\right)\left(-1+q^{16+24\,NN}\right)\left(-1+q^{18+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 9 OK

$$\left\{\frac{q^{-21-36\,NN}\left(-1+q^{16+24\,NN}+q^{18+24\,NN}+q^{20+24\,NN}-q^{34+48\,NN}-q^{36+48\,NN}-q^{38+48\,NN}+q^{54+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-21-36\,NN}\left(-1+q^{16+24\,NN}\right)\left(-1+q^{18+24\,NN}\right)\left(-1+q^{20+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 10 OK

$$\left\{\frac{q^{-24-36\,NN}\left(-1+q^{18+24\,NN}+q^{20+24\,NN}+q^{22+24\,NN}-q^{38+48\,NN}-q^{40+48\,NN}-q^{42+48\,NN}+q^{60+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-24-36\,NN}\left(-1+q^{18+24\,NN}\right)\left(-1+q^{20+24\,NN}\right)\left(-1+q^{22+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 11 OK

$$\left\{\frac{q^{-9\,(3+4\,NN)}\left(-1+q^{24\,(1+NN)}+q^{20+24\,NN}+q^{22+24\,NN}-q^{42+48\,NN}-q^{44+48\,NN}-q^{46+48\,NN}+q^{66+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-9\,(3+4\,NN)}\left(-1+q^{24\,(1+NN)}\right)\left(-1+q^{20+24\,NN}\right)\left(-1+q^{22+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 12 OK

$$\left\{\frac{q^{-6\,(5+6\,NN)}\left(-1+q^{24\,(1+NN)}-q^{48\,(1+NN)}+q^{72\,(1+NN)}+q^{22+24\,NN}+q^{26+24\,NN}-q^{46+48\,NN}-q^{50+48\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-6\,(5+6\,NN)}\left(-1+q^{24\,(1+NN)}\right)\left(-1+q^{22+24\,NN}\right)\left(-1+q^{26+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 13 OK

$$\left\{\frac{q^{-33-36\,NN}\left(-1+q^{24\,(1+NN)}+q^{26+24\,NN}+q^{28+24\,NN}-q^{50+48\,NN}-q^{52+48\,NN}-q^{54+48\,NN}+q^{78+72\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ \frac{q^{-33-36\,NN}\left(-1+q^{24\,(1+NN)}\right)\left(-1+q^{26+24\,NN}\right)\left(-1+q^{28+24\,NN}\right)}{\left(-1+q^{2}\right)^{3}\left(1+2\,q^{2}+2\,q^{4}+q^{6}\right)},\ 0\right\}$$

m = 14 OK

Vita

Eric Jacob was born in West Allis, Wisconsin. He has a B.S in Mechanical Engineering from the Milwaukee School of Engineering (MSOE) with a minor in Physics. He studied at the University of Tennessee Space Institute (UTSI) under Dr. Gary Flandro where he received a M.S. and PhD in Aerospace Engineering. In his time at UTSI he worked extensively with Dr. Boris Kupershmidt and assisted him in his research. The work presented in this thesis is a result of his time working with Dr. Kupershmidt.