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Lagrangian Representations of (p, p, p)-triangle Groups

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To the Graduate Council:

I am submitting herewith a dissertation written by Paul Wayne Lewis Jr. entitled "Lagrangian Representations of (p, p, p)-triangle Groups." I have examined the final electronic copy of this dissertation for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, with a major in Mathematics.

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Lagrangian Representations of (p, p, p)-triangle Groups

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Abstract

We obtain explicit formulae for Lagrangian representations of the (p, q, r)-triangle group into the group of direct isometries of the complex hyperbolic plane in the case where p=q=r. Numerically approximated matrix generators of representations of the (p, p, p)-triangle group are obtained using a special basis. The numerical approximations are then used to guess the exact generators by a process utilizing the LLL algorithm. The matrices are proved rigorously to generate Lagrangian representations of the (p, p, p)-triangle group and are applied to the problem of deciding whether or not an interval contains representations of the (p, p, p)-triangle group which are not discrete.

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Chapter 1

Introduction

The (p,q,r)-*triangle group* is the group $\Delta(p,q,r)$ with presentation

$$\Delta(p,q,r) \;=\; \langle a,b,c \mid a^p = b^q = c^r = abc = 1\rangle$$

It is well-known that for p,q,r satisfying $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ there exists a discrete, faithful representation ϕ of $\Delta(p,q,r)$ into $\mathrm{Isom}^+(\mathbb{R}H^2)$, the group of direct isometries of $\mathbb{R}H^2$, and that this representation is unique up to conjugacy. The group $\Delta(p,q,r)$ is a subgroup of index 2 of the *reflection triangle group* $\Delta_r(p,q,r)$ generated by reflections in the sides of a triangle in $\mathbb{R}H^2$ with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, $\frac{\pi}{r}$, and this reflection triangle group generates a tiling of the hyperbolic plane (see Fig. 1 for an example). We may view the generators a,b,c of $\Delta(p,q,r)$ as rotations through angles $\frac{2\pi}{p}$, $\frac{2\pi}{q}$, $\frac{2\pi}{r}$ about the respective vertices of the above triangle.

As stated above, the discrete, faithful representation of $\Delta(p,q,r)$ as isometries of $\mathbb{R}H^2$ is *rigid*, in the sense that it is unique up to conjugacy. However, we may compose this representation with the natural inclusion of $\mathrm{Isom}^+(\mathbb{R}H^2)$ into $\mathrm{Isom}^+(\mathbb{C}H^2)$, the group of direct isometries of the complex hyperbolic plane, and in general the resulting composite representation of $\Delta(p,q,r)$ into $\mathrm{Isom}^+(\mathbb{C}H^2)$ is no longer rigid. Let us denote this representation $\phi_0 : \Delta(p,q,r) \to \mathrm{Isom}^+(\mathbb{C}H^2)$. It is proved in [CG05] that if each of p,q,r is greater than 2, the representation ϕ_0 admits a 2-dimensional family $\{\phi_{u,v}\}$ of non-conjugate deformations, called the *Hitchin component* of the representation variety.

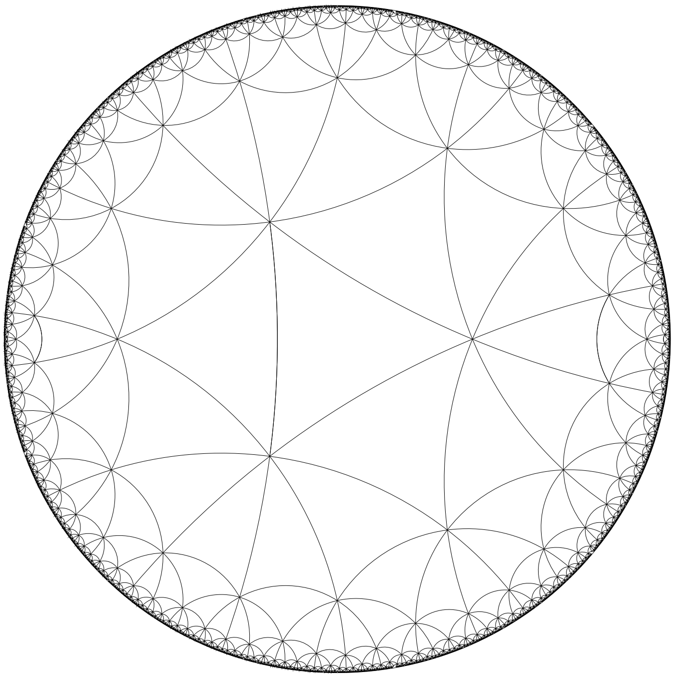

Fig.1: the tiling of $\mathbb{R}H^2$ *generated by* $\Delta_r(4,4,4)$

As in the real hyperbolic case, the fixed points of the generators a, b, c under $\phi_{u,v}$ are the vertices of a triangle in the complex hyperbolic plane $\mathbb{C}H^2$. For the "base" representation ϕ_0 of $\Delta(p,q,r)$, by construction these triangle vertices lie in the canonically embedded $\mathbb{R}H^2$ inside $\mathbb{C}H^2$, but for general u, v this is not the case. However, there does exist a curve of representations in the Hitchin component for which the three fixed points continue to lie in the canonically embedded $\mathbb{R}H^2$ in $\mathbb{C}H^2$, and this 1-dimensional subfamily of deformations of ϕ_0 is the topic of this thesis. We call the representations in this 1-dimensional subfamily *Lagrangian*, as the term *Lagrangian plane* is traditionally used to denote any subspace of $\mathbb{C}H^2$ that is the image of the canonically embedded $\mathbb{R}H^2$ under an isometry of $\mathbb{C}H^2$.

The main purpose of this thesis is to obtain explicit formulae for representations, and to see how these can be used for further investigations into such matters as discreteness.

It is a well-documented fact that computations in the realm of $\mathbb{C}H^2$ are usually extremely cumbersome and arduous. For this reason the strategic decision was made to restrict attention to the case $p = q = r$, where we note that the condition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ dictates that $p \geq 4$. Initially, even with this restriction, the search for a formula for the curve of Lagrangian representations of $\Delta(p, p, p)$ rapidly became bogged down in expressions for matrix entries taking up most of a page; the breakthrough came when it was decided to try discarding the conventional basis for the complex vector space $\mathbb{C}^3$ and substituting a new basis that arises naturally from the geometric context.

Let P_0 denote the canonical real hyperbolic plane embedded in $\mathbb{C}H^2$. Our starting point is to take a triangle in P_0, and to obtain a representation of the reflection triangle group $\Delta(p, p, p)$ by mapping the three generating involutions to (anti-holomorphic) *inversions* in Lagrangian planes P_1, P_2, P_3 meeting P_0 in the respective sides of the triangle. Any two of P_1, P_2, P_3 meet at a single point, namely a vertex of the triangle, and all are inclined at the same angle to P_0; we denote this angle θ. It is convenient to use θ as a parameter for the curve of Lagrangian representations. At the base representation ϕ_0, the parameter θ has the value $\frac{\pi}{2}$, and θ decreases as we move along the curve. Away from ϕ_0, the angles of the triangle are no longer $\frac{\pi}{p}$, but fortunately there is a simple relation between this angle and the two quantities p, θ.

In principle it should be possible to compute the formula for the curve of Lagrangian representations directly from the geometric setup using exact arithmetic, but this approach turns out to be impractical, due to the extremely cumbersome expressions that are encountered. Instead, numerical approximations to matrices were obtained to a high degree of accuracy using *Mathematica* [Wol10], and exact values for matrix entries were guessed using the celebrated Lenstra-Lenstra-Lovász (LLL) algorithm [LLL82]. This was done for a sequence of suitable values of the inclination angle θ and for fixed $p \in \{4, 5, 6\}$, and polynomial interpolation was used to guess a general formula for the matrix entries for this particular p. Finally, general formulae for the matrix entries were guessed covering all

$p \geq 4$ and all θ. Fortunately it was possible to guess the final formulae from a relatively small amount of experimental data. Some more details of this computation are given in Chapter 4.

Of course, since these computations were based on numerical approximations and several instances of guesswork, at this stage they are not rigorous. It is therefore essential to check, using exact computations, that the candidate matrices, whose entries are expressions in p and θ, really do give the desired Lagrangian representations of $\Delta(p, q, r)$. Although the matrices for the generators a, b, c of $\Delta(p, p, p)$ have reasonably pleasant entries, and although the necessary exact computations are elementary, involving only basic algebra and trigonometry, they are sufficiently complicated that we use *Mathematica* as an aid. It is important to emphasize that here *Mathematica* operates in *exact* mode, so rigor is not compromised.

The thesis is organized as follows. In Chapter 2 we provide background on the complex hyperbolic plane and other prerequisites. Chapter 3 explains how we obtain a representation of the reflection triangle group using inversions in the Lagrangian planes P_1, P_2, P_3, and Chapter 4 gives an account of how numerical approximations are obtained of the matrices representing the generators a, b, c of $\Delta(p, p, p)$, using the special basis for $\mathbb{C}^3$. The exact matrices are presented in Chapter 5, and the rigorous verification is given that they are correct. Chapter 6 contains an application to the important and difficult question of deciding whether a given representation of $\Delta(p, p, p)$ is discrete.

Chapter 2

The Complex Hyperbolic Plane

2.1 Hermitian Forms and $SU(2,1)$

We begin by giving an overview of some basic definitions and concepts concerning the complex hyperbolic plane, $\mathbb{C}H^2$.

Let A be an $n \times n$ matrix over the complex numbers. A is said to be *Hermitian* if A is equal to its conjugate transpose A^*, and A is *unitary* if $A^{-1} = A^*$. It follows immediately from the definitions that the eigenvalues of an Hermitian matrix are real, and that the eigenvalues of a unitary matrix lie on the unit circle of the complex plane. Each Hermitian matrix is unitarily diagonalizable, *i.e.* if J is Hermitian there exists a unitary matrix P such that $P^{-1}JP$ is diagonal. If J is a non-singular $n \times n$ Hermitian matrix with r positive eigenvalues and s negative eigenvalues ($r + s = n$), we say that the *signature* of J is the ordered pair (r, s).

The *Hermitian form* associated to an $n \times n$ Hermitian matrix J is the form $\langle\ ,\ \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ defined by $\langle \mathbf{z}\,,\, \mathbf{w} \rangle = \mathbf{w}^*\, J\, \mathbf{z}$, where $\mathbf{z}, \mathbf{w}$ are vectors in column form. If J has signature (r, s), we also say that the associated form $\langle\ ,\ \rangle$ has signature (r, s). The *standard form of signature* (r, s) is

$$\langle \mathbf{z}\,,\, \mathbf{w} \rangle = \overline{w_1}\, z_1 + \cdots + \overline{w_r}\, z_r - \overline{w_{r+1}}\, z_{r+1} - \overline{w_n}\, z_n \quad ,$$

induced by the diagonal matrix J with the first r entries on the diagonal being 1 and the remaining diagonal entries -1; any form of signature (r, s) is equivalent via a change of basis to this standard form. We use $\mathbb{C}^{r,s}$ to denote the vector space $\mathbb{C}^n$ endowed with the standard form of signature (r, s).

The concept of "unitary matrix" may be generalized as follows. If J is a non-singular, Hermitian, $n \times n$ matrix, then an $n \times n$ matrix A is *unitary with respect to the form* J if $A^{-1} = J^{-1}A^*J$, or equivalently $A^*JA = J$. The set of all $n \times n$ matrices that are unitary with respect to the standard (r, s) form forms a group denoted $U(r, s)$; the subgroup consisting of matrices of determinant 1 is denoted $SU(r, s)$. We also need to consider the *projective unitary group* $PU(r, s)$, defined to be the quotient of $U(r, s)$ by the subgroup of scalar matrices, *i.e.* the subgroup consisting of diagonal matrices whose diagonal entries are all $e^{i\theta}$ for some fixed θ.

Henceforth we will be concerned exclusively with forms of signature $(2, 1)$ on the 3-dimensional complex vector space $\mathbb{C}^3$. We note that we can regard $PU(2, 1)$ either as the quotient of $U(2, 1)$ by the subgroup consisting of all $e^{i\theta}I$, or as the quotient of $SU(2, 1)$ by the subgroup of order 3 generated by

$$\begin{bmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix},$$

where $\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity. Thus we see that $SU(2, 1)$ is a 3-fold cover of its quotient $PU(2, 1)$.

2.2 The Projective Model of $\mathbb{C}H^2$

Let $\mathbb{C}^{2,1}$ be $\mathbb{C}^3$ equipped with the Hermitian form:

$$\langle \boldsymbol{x}, \boldsymbol{z} \rangle = x_1\overline{z_1} + x_2\overline{z_2} - x_3\overline{z_3}$$

where $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^{2,1}$ and $\boldsymbol{x} = [x_1, x_2, x_3]^T$ and $\boldsymbol{y} = [z_1, z_2, z_3]^T$.

We observe that for any $\lambda \in \mathbb{C}$,

$$\langle \lambda \boldsymbol{x}, \lambda \boldsymbol{z} \rangle = |\lambda|^2 \langle \boldsymbol{x}, \boldsymbol{z} \rangle \ ,$$

and that

$$\begin{aligned} \langle \boldsymbol{z}, \boldsymbol{z} \rangle &= z_1 \overline{z_1} + z_2 \overline{z_2} - z_3 \overline{z_3} \\ &= |z_1|^2 + |z_2|^2 - |z_3|^2 \ , \end{aligned}$$

whence $\langle \boldsymbol{z}, \boldsymbol{z} \rangle \in \mathbb{R}$. We call a vector $\boldsymbol{z} \in \mathbb{C}^{2,1}$ positive, null, or negative when $\langle \boldsymbol{z}, \boldsymbol{z} \rangle > 0$, $\langle \boldsymbol{z}, \boldsymbol{z} \rangle = 0$, or $\langle \boldsymbol{z}, \boldsymbol{z} \rangle < 0$, respectively. From the identity $\langle \lambda \mathbf{z}\, ,\, \lambda \mathbf{z} \rangle = |\lambda|^2 \langle \mathbf{z}\, ,\, \mathbf{z} \rangle$, we see that the property of being positive, null or negative is unaffected by multiplication by a non-zero scalar. Therefore the property holds for all non-zero vectors in a 1-dimensional subspace, and it makes sense to speak of *positive, null or negative lines* in $\mathbb{C}^{2,1}$.

Definition 1. The projective model of $\mathbb{C}H^2$ is the collection of negative lines in $\mathbb{C}^{2,1}$. The boundary of the projective model is the collection of null lines in $\mathbb{C}^{2,1}$.

$\mathbb{C}H^2$ is a space of four (real) dimensions with $\partial \mathbb{C}H^2$ homeomorphic to S^3.

The set of negative lines in $\mathbb{C}H^2$ with real coordinates is a 2-dimensional subspace, which we regard as a naturally embedded copy of the real hyperbolic place $\mathbb{R}H^2$. This subspace is the Lagrangian plane P_0 mentioned in the Introduction.

When working in the projective model of $\mathbb{C}H^2$, it is customary (with slight abuse of notation) to denote a line by any non-zero (column) vector contained in that line. Thus if λ is a non-zero complex number we may write

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \lambda \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \lambda z_1 \\ \lambda z_2 \\ \lambda z_3 \end{bmatrix} \quad .$$

If $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, then it must be the case that $z_3 \neq 0$. Hence, every point in $\mathbb{C}H^2$ can be represented by a vector of the form

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}$$

Other Hermitian forms of signature $(2, 1)$ may be used instead of the one stated above. A useful example is the Hermitian form

$$\langle \boldsymbol{x}, \boldsymbol{z} \rangle = x_1 \overline{z_3} + x_2 \overline{z_2} + x_3 \overline{z_1}$$

This form gives rise to the Siegel domain model of $\mathbb{C}H^2$. One important aspect of the Siegel domain model is its usefulness in placing coordinates on the boundary of $\mathbb{C}H^2$.

2.3 Metric and Sectional Curvature

We now state the metric on the projective model of $\mathbb{C}H^2$:

Definition 2. The Bergman metric, $\rho(\boldsymbol{x}, \boldsymbol{z})$, on the projective model of $\mathbb{C}H^2$ is

$$\cosh^2\left(\frac{\rho(\boldsymbol{x}, \boldsymbol{z})}{k}\right) = \frac{\langle \boldsymbol{x}, \boldsymbol{z} \rangle \langle \boldsymbol{z}, \boldsymbol{x} \rangle}{\langle \boldsymbol{x}, \boldsymbol{x} \rangle \langle \boldsymbol{z}, \boldsymbol{z} \rangle}$$

where $k = 1$ or $k = 2$, depending on convention.

The value of the above expression is unchanged if we replace $\mathbf{x}, \mathbf{z}$ by non-zero scalar multiples thereof; therefore it is well-defined on lines.

Regarding the choice of k, some major references use $k = 2$, for example [Gol99]. For our purposes, we choose $k = 1$; this will guarantee that Lagrangian subspaces (in particular P_0) have constant sectional curvature -1. The sectional curvature of $\mathbb{C}H^2$ is not constant, but lies in the interval $[-4, -1]$. The non-constant property of the sectional curvature of $\mathbb{C}H^2$ is the source of much technical difficulty, but also is the source of interesting features.

2.4 Isometries of $\mathbb{C}H^2$

We now describe the isometries of $\mathbb{C}H^2$.

Theorem 1. *The matrix group $PU(2,1)$ acts as holomorphic isometries on the projetive model of $\mathbb{C}H^2$. All anti-holomorphic isometries are given by complex conjugation followed by an element of $PU(2,1)$.*

Instead of working with $PU(2,1)$ directly, it is more practical to work in its triple cover $SU(2,1)$. Thus an element of $PU(2,1)$ can be denoted by any one of its three preimages in $SU(2,1)$, but this does not cause any problems.

Isometries of $\mathbb{C}H^2$ can be classified according to their fixed points.

Definition 3. A complex hyperbolic isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbb{C}H^2$, *parabolic* if it fixes exactly one point of $\partial\mathbb{C}H^2$, or *elliptic* if it fixes some point in $\mathbb{C}H^2$. An elliptic isometry is *regular elliptic* if and only if it has distinct eigenvalues.

Let $A \in SU(2,1)$. Then one may determine if the isometry A is loxodromic, parabolic, or elliptic by examining $tr(A)$.

Theorem 2. *Let $A \in SU(2,1)$, and let $\varphi = tr(A)$. Also, let f be the function*

$$f(z) = |z|^4 - 8\mathfrak{Re}\left(z^3\right) + 18|z|^2 - 27$$

then A is regular elliptic if and only if $f(\varphi) < 0$.

$f^{-1}\{0\}$ is a deltoid curve (see Fig. 2). Points on the deltoid represent conjugacy classes of parabolic elements. Elements whose trace lies inside the deltoid are elliptic, and those with traces outside the deltoid are loxodromic.

Corollary 1. *$A \in SU(2,1)$ is regular elliptic with $tr(A) \in \mathbb{R}$ if and only if $tr(A) \in (-1,3)$.*

Lemma 1. *If $A \in SU(2,1)$, A is regular elliptic and $tr(A) \in \mathbb{R}$, then A has eigenvalues $e^{i\phi}$, $e^{-i\phi}$, and 1 where $\cos\theta = tr(A) - 1$.*

The proof of the above lemma is found in [Par04], Theorem 3.6.

Lemma 2. *Let $A \in SU(2,1)$ be regular elliptic with real trace. If A has order p, then A has eigenvalues $e^{\frac{2\pi}{p}}$, $e^{-\frac{2\pi}{p}}$, and 1.*

Proof. Following the previous lemma, suppose the eigenvalues of A are $e^{i\phi}$, $e^{-i\phi}$, and 1. We know ϕ cannot be an integer multiple of π, because A regular elliptic implies each eigenvalue is distinct. Suppose without loss of generality that $\phi \in (0, 2\pi)$. Since the eigenvalues of A are distinct, A is diagonalizable. So, $A = PDP^{-1}$ where D is a diagonal matrix with entries $e^{i\phi}$, $e^{-i\phi}$, and 1. Thus,

$$A^p = \left(PDP^{-1}\right)^p = PD^pP^{-1} = I$$

Thus, $D^p = I$, and hence $\left(e^{i\phi}\right)^p = 1$. So, $\phi = \frac{2\pi}{p}$. □

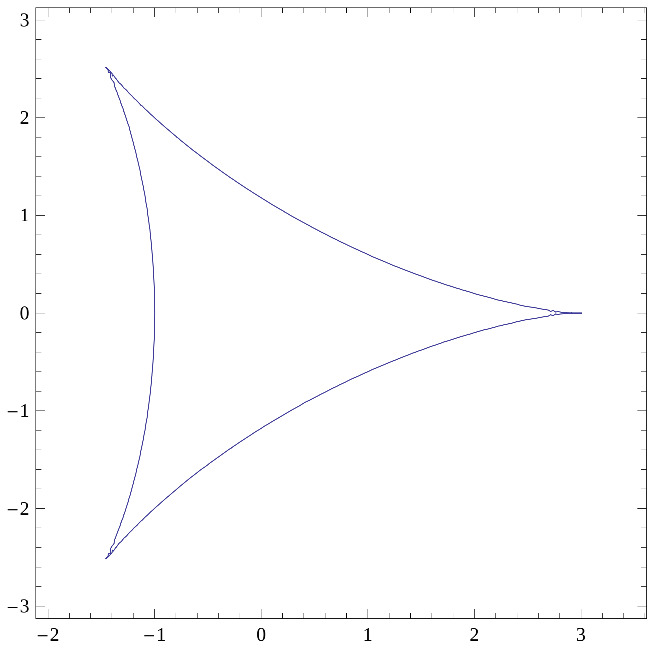

Fig.2: The deltoid $|z|^4 - 8\mathfrak{Re}\left(z^3\right) + 18|z|^2 - 27 = 0$

2.5 Lagrangian Planes in $\mathbb{C}H^2$

Again, we will represent by P_0 the set $\{[x,y,1]^T | x,y \in \mathbb{R}\}$ in $\mathbb{C}H^2$. P_0 is an embedding of $\mathbb{R}H^2$ in $\mathbb{C}H^2$. We call this the *standard embedding of* $\mathbb{R}H^2$ *in* $\mathbb{C}H^2$.

Definition 4. A Lagrangian plane in $\mathbb{C}H^2$ is the image of P under some element of $PU(2,1)$.

Lagrangian planes are totally geodesic subspaces of $\mathbb{C}H^2$. The are the only totally real, totally geodesic subspaces of two real dimensions.

Definition 5. Let P be a Lagrangian plane. Reflection (or inversion) in P is the unique anti-holomorphic isometry that fixes P point-wise.

Let P be a Lagrangian plane. If we wish to find the inversion in P, i_P, we first identify an isometry, ψ, such that $\psi(P) = P_0$. Now, let c represent the isometry produced by conjugation, that is

$$c\left(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) = \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \\ \overline{z_3} \end{bmatrix}$$

Then $i_{P_0} = \psi^{-1} c \psi$.

Langrangian planes have the following connection with the full isometry group of $\mathbb{C}H^2$:

Theorem 3. *Every isometry of* $\mathbb{C}H^2$ *can be written as a product of at most three reflections in Lagrangian planes.*

2.6 The Hermitian Triple Product

Definition 6. Given a Hermitian form, the Hermitian triple product is defined to be

$$\langle \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \rangle := \langle \boldsymbol{z}_1, \boldsymbol{z}_2 \rangle \langle \boldsymbol{z}_2, \boldsymbol{z}_3 \rangle \langle \boldsymbol{z}_3, \boldsymbol{z}_1 \rangle$$

for bz_1, bz_2, $bz_3 \in \mathbb{C}^{2,1}$.

The triple product has been used to define the Brehm shape invariant for triangles and Cartan's angular invariant on $\partial\mathbb{C}H^2$.

Chapter 3

The Curve of Representations

3.1 Overview

The purpose of this chapter is to construct the curve of Lagrangian representaions of the (p,p,p)-triangle group into the group of holomorphic isometries of $\mathbb{C}H^2$. The work in this and subsequent chapters will be general, in that our constructions will hold for all $p \geq 4$.

Let $\triangle_p$ denote the abstract (p,p,p)-triangle group. We will construct a curve of representations

$$\Gamma_\theta \colon \triangle_p \to SU\,(2,1)$$

depending on $\theta \in \mathbb{R}$ and $p \geq 4$. Each map will be defined by indicating the image of each generator of $\triangle_p$. The image of the group in $SU\,(2,1)$ will be deformed by continuously varying θ while maintaining the necessary group relations.

Recall that a reflection in a Langrangian plane is an anti-holomorphic isometry of order two. It is not unreasonable to hope that, under suitable conditions, reflections in Lagrangian planes can be used to generate reflection triangle groups, indeed, triangle groups. This is the approach we use. Recall from the Introduction that for each $p \geq 4$ there is a discrete faithful representation of $\triangle_p$ into the group of direct isometries of $\mathbb{R}H^2$, and that from the results of [CG05] this representation may be deformed to representations into the group of direct isometries of $\mathbb{C}H^2$. Given $p \geq 4$, we start with a base triangle in the standard embedding of $\mathbb{R}H^2$ with angles measuring $\frac{\pi}{p}$, and with its center at the origin, $[0,0,1]^T \in \mathbb{C}^{2,1}$. Later in this chapter, as promised in the Introduction, we shall specify Lagrangian planes P_1, P_2, P_3 meeting P_0 in the respective sides of the triangle, and having the property that any two meet precisely at a triangle vertex.

Let i_1, i_2, i_3 be the (anti-holomorphic) inversions in P_1, P_2, P_3 respectively, and let

$$a = i_1 i_2 \quad , \quad b = i_2 i_3 \quad , \quad c = i_3 i_1 \quad .$$

We will show that for suitable base triangle and suitable Lagrangian planes P_1, P_2, P_3, the group relations $a^p = c^p = c^p = abc = 1$ are satisfied, whereby a, b, and c generate a group of isometries isomorphic to $\triangle_p$. Note that a, b, and c each fix a vertex of the base triangle, and thus each of these generators is an elliptic isometry.

3.2 Constructing the Curve of Representations

We will now construct the initial representation of the (p,p,p)-triangle group. We begin by constucting an equilateral base triangle which lies in the standard embedding of $\mathbb{R}H^2$ in $\mathbb{C}H^2$ and is centered at the origin. For the group $\triangle_p$, all of the angles of the base triangle will initially be $\frac{\pi}{p}$.

Consider first the geodesic $\gamma = \{[x,0,1]^T | x \in \mathbb{R}\}$, which we call the "$x$-axis." We may translate γ in the negative y-direction by applying a real hyperbolic translation of form

$$t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh u & -\sinh u \\ 0 & -\sinh u & \cosh u \end{bmatrix} \quad .$$

The image of γ under this translation forms one of the sides of our base triangle. The other two sides are then obtained by rotating $t(\gamma)$ within P_0 about the origin through angles $\frac{2\pi}{3}$, $\frac{4\pi}{3}$ respectively. Rotation through $\frac{2\pi}{3}$ about the origin is achieved by the isometry

$$r = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The base triangle is then comprised of segments of geodesics $t\,(\gamma)$, $r\,t\,(\gamma)$, and $r^2\,t\,(\gamma)$. The values of $\sinh u$ and $\cosh u$ in the translation t above obviously depend on the angles of the the triangle. The following proposition gives the needed values for $\cosh u$ and $\sinh u$.

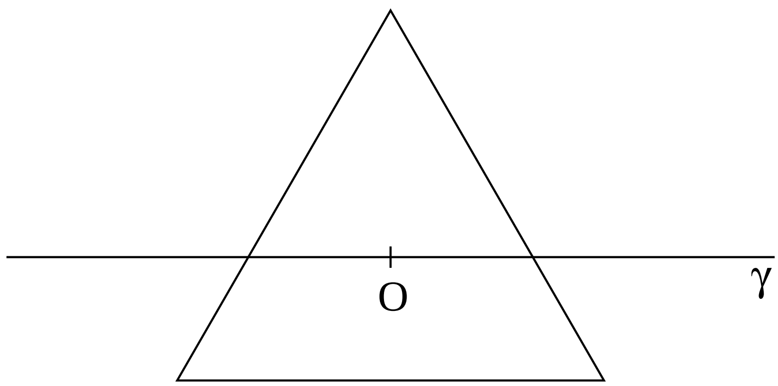

Fig.3: The position of the base triangle in relation to the x–axis γ

Proposition 1. *Let r, t, and γ be as above. Then the points of intersection among $t\,(\gamma)$, $r\,t\,(\gamma)$, and $r^2\,t\,(\gamma)$ are vertices of an equilateral triangle centered at the origin with each angle measuring $0 \le \omega < \frac{\pi}{3}$ when*

$$\sinh u = \sqrt{\frac{2\cos\omega - 1}{3}}, \cosh u = \sqrt{\frac{2\cos\omega + 2}{3}}$$

where $0 \le \omega < \frac{\pi}{3}$ is the desired measure of the angles of the triangle.

Proof. First, note that the projective model of $\mathbb{R}H^2$ is conformal at the origin, so the hyperbolic and Euclidean measures for any angle in P with vertex $[0,0,1]^T$ will coincide. Define an angle in P using the set of points $\{[0,0,1]^T, [x,0,1]^T, [x,y,1]^T \,|x,y \in \mathbb{R}, x < 0, y > 0\}$ with $[0,0,1]^T$ being the vertex of the angle. For simplicity in computation, choose x and y such that the determined angle has measure $\psi = \frac{\omega}{2}$ for some $0 \leq \omega < \frac{\pi}{3}$. Since the triangle to be constructed is centered at the origin, all vertices will be at an equal distance (Euclidean and hyperbolic) from the origin. Hence, the triangle will appear as a Euclidean triangle in P. Thus, we want each angle of the triangle to have Euclidean measure $\frac{\pi}{3}$. We will use the hyperbolic translation

$$T = \begin{bmatrix} \cosh v & 0 & \sinh v \\ 0 & 1 & 0 \\ \sinh v & 0 & \cosh v \end{bmatrix}$$

which fixes the x-axis to move the angle away from the origin while fixing the x-axis setwise until its Euclidean measure is $\frac{\pi}{6}$, that is, half of the ultimately desired measure of $\frac{\pi}{3}$. Since T is an isometry on P_0, the hyperbolic measure of the angle will remain constant. Acting by matrix multiplication, the effect of T on each of the vertices is the following:

$$T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sinh v \\ 0 \\ \cosh v \end{bmatrix} = \begin{bmatrix} \tanh v \\ 0 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x\cosh v + \sinh v \\ 0 \\ x\sinh v + \cosh v \end{bmatrix} = \begin{bmatrix} \frac{x+\tanh v}{x\tanh v+1} \\ 0 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x\cosh v + \sinh v \\ y \\ x\sinh v + \cosh v \end{bmatrix} = \begin{bmatrix} x + \tanh v \\ \frac{y}{\cosh t} \\ x\tanh t + 1 \end{bmatrix} = \begin{bmatrix} \frac{x+\tanh v}{x\tanh v+1} \\ \frac{y}{\cosh v + x\sinh v} \\ 1 \end{bmatrix}$$

T fixes the x-axis setwise, so one ray of the angle remains on the x-axis. Now we calculate the Euclidean angle, ϕ, defined by the images of the three points. We do this by representing the sides of the angle as vectors in $\mathbb{R}^2$. Since one ray lies in the x-axis, one vector is

$$\overline{a} = (-1, 0)$$

By subtracting coordinates in the images above, the other vector is given by

$$\overline{b} = \left(\frac{x + \tanh v}{x \tanh v + 1} - \tanh v, \frac{y}{\cosh v + x \sinh v} \right)$$

Let the first component of $\overline{b}$ be denoted A, and let the second component be denoted B. Using the familiar formula for calculating the angle between two Euclidean vectors and the fact that we want $\phi = \frac{\pi}{6}$,

$$\frac{\sqrt{3}}{2} = \cos \phi = \frac{\overline{a} \cdot \overline{b}}{\|\overline{a}\| \|\overline{b}\|} = \frac{-A}{\sqrt{A^2 + B^2}}$$

Squaring the above equation,

$$\frac{3}{4} = \cos^2 \phi = \frac{A^2}{A^2 + B^2} \Rightarrow 3B^2 = A^2$$

Thus,

$$
\begin{aligned}
3\frac{y^2}{(\cosh v + x\sinh v)^2} &= \left(\frac{x\tanh^2 v - x}{x\tanh v + 1}\right)^2 \\
\Rightarrow 3\frac{\frac{y^2}{\cosh^2 v}}{(1 + x\tanh v)^2} &= \frac{\left(x\tanh^2 v - x\right)^2}{(x\tanh v + 1)^2} \\
\Rightarrow \frac{3y^2}{\cosh^2 v} &= x^2\left(\tanh^2 v - 1\right)^2 \\
\Rightarrow \frac{3y^2}{x^2} &= \cosh^2 v\left(\tanh^2 v - 1\right)^2 \\
\Rightarrow \frac{\frac{3y^2}{x^2+y^2}}{\frac{x^2}{x^2+y^2}} &= \cosh^2 v\left(1 - \tanh^2 v\right)^2 \\
\Rightarrow \frac{3\sin^2\psi}{\cos^2\psi} &= \cosh^2 v\ \mathrm{sech}^2 v\left(1 - \tanh^2 v\right) \\
\Rightarrow 3\tan^2\psi &= 1 - \tanh^2 v \\
\Rightarrow \tanh^2 v &= 1 - 3\tan^2\psi \\
\Rightarrow \tanh v &= \sqrt{1 - 3\tan^2\psi}
\end{aligned}
$$

The point $[\tanh v, 0, 1]^T$ as calculated above is the correct distance from the origin to be a vertex for the triangle, but, in order for the triangle to have the appropriate orientation to utilize t, we rotate clockwise $\frac{\pi}{6}$.

$$
\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \tanh v \\ 0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} \frac{\sqrt{3}}{2}\tanh v \\ -\frac{1}{2}\tanh v \\ 1 \end{bmatrix}
$$

Since

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh u & -\sinh u \\ 0 & -\sinh u & \cosh u \end{bmatrix}
\begin{bmatrix} x \\ 0 \\ 1 \end{bmatrix}
=
\begin{bmatrix} x \\ -\sinh u \\ \cosh u \end{bmatrix}
=
\begin{bmatrix} \frac{x}{\cosh u} \\ -\tanh u \\ 1 \end{bmatrix}
$$

we know

$$
\begin{aligned}
\tanh u &= \frac{\tanh v}{2} \\
&= \frac{\sqrt{1 - 3\tan^2 \psi}}{2} \\
&= \frac{\sqrt{1 - 3\tan^2 \frac{\theta}{2}}}{2} \\
&= \frac{\sqrt{1 - 3\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)^2}}{2} \\
&= \frac{\sqrt{1 - \frac{3-3\cos\theta}{1+\cos\theta}}}{2} \\
&= \sqrt{\frac{1}{4} - \frac{3 - 3\cos\theta}{4 + 4\cos\theta}} \\
&= \sqrt{\frac{1 + \cos\theta}{4 + 4\cos\theta} - \frac{3 - 3\cos\theta}{4 + 4\cos\theta}} \\
&= \sqrt{\frac{2\cos\theta - 1}{2\cos\theta + 2}}
\end{aligned}
$$

Thus using hyperbolic trigonometric identities,

$$
\begin{aligned}
\tanh^2 u &= 1 - \frac{1}{\cosh^2 u} \\
&\Rightarrow \frac{2\cos\theta - 1}{2\cos\theta + 2} = 1 - \frac{1}{\cosh^2 u} \\
&\Rightarrow \frac{-3}{2\cos\theta + 2} = \frac{1}{\cosh^2 u} \\
&\Rightarrow \cosh^2 u = \frac{2\cos\theta + 2}{3} \\
&\Rightarrow \cosh u = \sqrt{\frac{2\cos\theta + 2}{3}}
\end{aligned}
$$

and,

$$
\begin{aligned}
\cosh^2 u - \sinh^2 u = 1 & \\
& \Rightarrow \frac{2\cos\theta + 2}{3} - \sinh^2 u = 1 \\
& \Rightarrow \frac{2\cos\theta - 1}{3} - \sinh^2 u = 0 \\
& \Rightarrow \frac{2\cos\theta - 1}{3} = \sinh^2 u \\
& \Rightarrow \sinh u = \sqrt{\frac{2\cos\theta - 1}{3}}
\end{aligned}
$$

□

We turn now to construction of the Lagrangian planes P_1, P_2, P_3. Recall that P_0 represents the standard embedding of $\mathbb{R}H^2$ in $\mathbb{C}H^2$, that is $\{[x, y, 1]^T \mid x \in \mathbb{R}\}$. Clearly, P_0 contains the geodesic γ. We may obtain a one-parameter family of Lagrangian planes containing γ by applying to P the isometry

$$
s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

where $0 \leq \theta \leq \frac{\pi}{2}$. Then, for each $\theta \in \left[0, \frac{\pi}{2}\right]$, $\{[x, ye^{i\theta}, 1]^T \mid x, y \in \mathbb{R}\}$ is a Lagrangian plane containing γ. One-parameter families of Lagrangian planes containing each of the three sides of the base triangle are obtained by applying t, rt, and r^2t to $\{[x, ye^{i\theta}, 1]^T | x, y \in \mathbb{R}\}$. We will denote each of these Lagrangian planes as P_1, P_2, and P_3, repectively. From this point forward θ will be referred to as the *angle of inclination* of the Langrangian planes.

The Langrangian representations of $\triangle_p = \langle a, b, c \mid a^p = b^p = c^p = abc = 1\rangle$ into $PU(2,1)$ are constucted using inversions in these Lagrangian planes P_1, P_2, P_3. We will denote inversions in P_1, P_2, and P_3 by i_1, i_2, and i_3, respectively, and let $a = i_1 i_2$, $b = i_2 i_3$, and $c = i_3 i_1$.

For the initial representation we consider the Lagrangian planes containing the sides of the base triangle that are "perpendicular" to P_0, that is, obtained using $\theta = \frac{\pi}{2}$. The representation will be deformed into $SU(2,1)$ by continuously decreasing the angle of inclination, θ. As the angle of inclination decreases, the base triangle in the standard embedding of $\mathbb{R}H^2$ will necessarily deform in order for the group relations to be preserved. The theorem that follows describes the relationship between θ and the angles of the base triangle that is required to preserve the group relations of $\triangle_p$.

Theorem 4. *Let α be the measure of the angles of the base triangle in the standard embedding of $\mathbb{R}H^2$ when the Lagrangian planes have angle of inclination θ. Then $a = i_1 i_2$, $b = i_2 i_3$, and $c = i_3 i_1$ generate a representation of the (p, p, p) triangle group when*

$$\sin(\alpha)\sin(\theta) = \sin\left(\frac{\pi}{p}\right)$$

Proof. Without loss of generality, suppose we have an equilateral base triangle with angles measuring $\frac{\pi}{p}$ such that one vertex lies at the origin, another vertex lies on the positive x-axis, and all three vertices are contained in P. Suppose also that P_1, P_2, and P_3 are Langrangian planes, each containing one side of the base triangle so that P_1 and P_2 intersect at the origin, with P_1 containing the x-axis. Reflection in P_1 is given by scs^{-1} where c is reflection in P. Similarly, reflection in P_2 is given by $qscs^{-1}q^{-1}$ where q is the rotation

$$q = \begin{bmatrix} \cos\frac{\pi}{p} & \sin\frac{\pi}{p} & 0 \\ -\sin\frac{\pi}{p} & \cos\frac{\pi}{p} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As maps on $\mathbb{C}^{2,1}$ reflections in P_1 and P_2 are given by

$$i_1\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{x} \\ \overline{y}e^{2i\theta} \\ \overline{z} \end{bmatrix}$$

$$i_2\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow \begin{bmatrix} \overline{x}\cos^2\alpha + \overline{y}\sin\alpha\cos\alpha + e^{2i\theta}\left(\overline{x}\sin^2\alpha + \overline{y}\sin\alpha\cos\alpha\right) \\ \overline{y}\sin^2\alpha + \overline{x}\sin\alpha\cos\alpha + e^{2i\theta}\left(-\overline{x}\sin\alpha\cos\alpha + \overline{y}\cos^2\alpha\right) \\ \overline{z} \end{bmatrix}$$

Hence,

$$\begin{aligned} i_1 i_2\colon \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\longrightarrow \begin{bmatrix} x\cos^2\alpha + y\sin\alpha\cos\alpha + e^{-2i\theta}\left(x\sin^2\alpha + y\sin\alpha\cos\alpha\right) \\ e^{2i\theta}\left(y\sin^2\alpha + x\sin\alpha\cos\alpha\right) - x\sin\alpha\cos\alpha + y\cos^2\alpha \\ z \end{bmatrix} \\ &= \begin{bmatrix} x\left(\cos^2\alpha + e^{-2i\theta}\sin^2\alpha\right) + y\sin\alpha\cos\alpha\left(1 - e^{-2i\theta}\right) \\ x\left(e^{2i\theta} - 1\right)\sin\alpha\cos\alpha + y\left(e^{2i\theta}\sin^2\alpha + \cos^2\alpha\right) \\ z \end{bmatrix} \end{aligned}$$

Thus, the composition of the reflections in P_1 and P_2, which is a holomorphic isometry of $\mathbb{C}H^2$, may be represented by the matrix

$$i_1 i_2 = \begin{bmatrix} \cos^2\alpha + e^{-2i\theta}\sin^2\alpha & \sin\alpha\cos\alpha\left(1 - e^{-2i\theta}\right) & 0 \\ \left(e^{2i\theta} - 1\right)\sin\alpha\cos\alpha & e^{2i\theta}\sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$
\begin{aligned}
tr\,(i_1 i_2) &= \cos^2\alpha + e^{-2i\theta}\sin^2\alpha + e^{2i\theta}\sin^2\alpha + \cos^2\alpha + 1 \\
&= 2\cos^2\alpha + \left(e^{-2i\theta} + e^{2i\theta}\right)\sin^2\alpha + 1 \\
&= 2\cos^2\alpha + 2\cos(2\theta)\sin^2\alpha + 1
\end{aligned}
$$

When $\theta = \frac{\pi}{2}$ and $\alpha = \frac{\pi}{p}$,

$$
\begin{aligned}
tr\,(i_1 i_2) &= 2\cos^2\alpha + 2\cos(2\theta)\sin^2\alpha + 1 \\
&= 2\cos^2\left(\frac{\pi}{p}\right) + 2\cos(\pi)\sin^2\left(\frac{\pi}{p}\right) + 1 \\
&= 2\cos^2\left(\frac{\pi}{p}\right) - 2\sin^2\left(\frac{\pi}{p}\right) + 1 \\
&= 2\cos\left(\frac{2\pi}{p}\right) + 1
\end{aligned}
$$

In fact, each matrix representing $i_1 i_2$ in the deformation must have the same trace since $(i_1 i_2)^p = 1$. Clearly, $i_1 i_2$ is elliptic for all $\theta, \alpha \in \left[0, \frac{\pi}{2}\right]$ since the origin, $[0,0,1]^T$, is fixed. Also, $tr\,(i_1 i_2) \in \mathbb{R}\ \forall\ \theta, \alpha \in \left[0, \frac{\pi}{2}\right]$. Thus, using Lemma 1, each matrix has has distinct eigenvalues $1, e^{-i\phi}, e^{i\phi}$ for some $\phi \in (0, 2\pi)$. Therefore, each matrix is diagonalizable. Thus, in order for the group relations to be maintained, one needs $\left(e^{i\phi}\right)^p = 1$. So, $\phi = \frac{2\pi}{p}$, and $tr\left(i_1 i_2\right) = e^{-i\frac{2\pi}{p}} + e^{i\frac{2\pi}{p}} + 1 = 2\cos\left(\frac{2\pi}{p}\right) + 1$. Therefore,

$$\begin{aligned}
&2\cos^2\alpha + 2\cos\left(2\theta\right)\sin^2\alpha + 1 = 2\cos\left(\frac{2\pi}{p}\right) + 1\\
&\Rightarrow 2\cos^2\alpha + 2\cos\left(2\theta\right)\sin^2\alpha = 2\cos\left(\frac{2\pi}{p}\right)\\
&\Rightarrow \left(1-\sin^2\alpha\right) + \left(1-2\sin^2\theta\right)\left(1-\cos^2\alpha\right) = 1 - 2\sin^2\frac{\pi}{p}\\
&\Rightarrow 1-\sin^2\alpha + 1 + 2\sin^2\theta\cos^2\alpha - 2\sin^2\theta - \cos^2\alpha = 1 - 2\sin^2\frac{\pi}{p}\\
&\Rightarrow 1 + 2\sin^2\theta\cos^2\alpha - 2\sin^2\theta = 1 - 2\sin^2\frac{\pi}{p}\\
&\Rightarrow 2\sin^2\theta - 2\sin^2\theta\cos^2\alpha = 2\sin^2\frac{\pi}{p}
\end{aligned}$$

So,

$$\begin{aligned}
\sin^2\frac{\pi}{p} &= \sin^2\theta - \sin^2\theta\cos^2\alpha\\
&= \sin^2\theta\left(1-\cos^2\alpha\right)\\
&= \sin^2\theta\sin^2\alpha
\end{aligned}$$

The result follows since θ, α, and $\frac{\pi}{p}$ belong to $\left[0, \frac{\pi}{2}\right]$. □

The above construction can be generalized for constructing Langrangian representations for general (p, q, r)-triangle groups, using an inital base triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$. In this more general scenario, one would discover the deformation to be governed by the equations

$$
\begin{aligned}
\sin\alpha\sin\theta &= \sin\frac{\pi}{p}\\
\sin\beta\sin\theta &= \sin\frac{\pi}{q}\\
\sin\gamma\sin\theta &= \sin\frac{\pi}{r}
\end{aligned}
$$

where α, β, and γ are the angles of the base triangle with initial measures $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$, respectively.

Also, a consequence of Theorem 4 is that the deformation only has geometric significance for θ such that $\sin\theta > \frac{2}{\sqrt{3}}\sin\frac{\pi}{p}$. This proceeds from the requirement that since the base triangle lies in P, it must be that $3\alpha < \pi$. One would expect the base triangle to be degenerate when $\sin\theta = \frac{2}{\sqrt{3}}\sin\frac{\pi}{p}$. As the triangle degenerates to a point, its angles approach $\frac{\pi}{3}$. We define θ_p to be the value of the angle of inclination for which the base triangle becomes degenerate. That is, $\theta_p = \sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{p}\right)$.

Chapter 4

Constructing Normalized Representations

4.1 Generators and Eigenvectors

The purpose of this chapter is to lay the groundwork for constructing formulas for generators of Lagrangian representations of (p, p, p)-triangle groups using the vertices of the base triangle as a basis for these generators. We will first outline some important connections between eigenvectors and generators. Then, the properties that will characterize our generators, yielding a canonical form, will be discussed. The unveiling of the specific generators that give rise to a curve of Lagrangian representations of $\triangle_p$ for each $p \geq 4$ will occur in the next chapter, as well as the proof that these generators are as claimed.

Now, we start the process of demonstrating the desired properties of our generators by exploring further the connections between isometries and eigenvectors. The generators of our triangle group each leave fixed a vertex of the base triangle. Thus, the generators are each elliptic. Fixed points of isometries have a nice characterization.

Lemma 3. *Let $A \in SU(2,1)$. Then $\boldsymbol{x} \in \mathbb{C}^{2,1}$ represents a fixed point of A in $\mathbb{C}H^2$ if and only if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ and $\boldsymbol{x}$ is an eigenvector of A.*

Proof. Suppose that $\boldsymbol{x} \in \mathbb{C}^{2,1}$ represents a fixed point of A in $\mathbb{C}H^2$. Then $\boldsymbol{x}$, $A\boldsymbol{x}$ represent the same point of $\mathbb{C}H^2$. Since the points of $\mathbb{C}H^2$ are 1-dimensional subspaces of $\mathbb{C}^{2,1}$, we deduce that $\boldsymbol{x}$, $A\boldsymbol{x}$ are two non-zero vectors on the same line. Therefore $A\boldsymbol{x}$ is a non-zero scalar multiple of $\boldsymbol{x}$, and so $\boldsymbol{x}$ is an eigenvector of A. Also, since $\boldsymbol{x}$ represents a point of $\mathbb{C}H^2$, by definition $\langle \boldsymbol{x} , \boldsymbol{x} \rangle < 0$.

Suppose that $\langle \boldsymbol{x} , \boldsymbol{x} \rangle < 0$ and $\boldsymbol{x}$ is an eigenvector of A. Then, by definition, $\boldsymbol{x}$ represents a point of $\mathbb{C}H^2$. Since $A\boldsymbol{x} = \lambda \boldsymbol{x}$ for some non-zero complex number λ, the vectors $\boldsymbol{x}$, $A\boldsymbol{x}$ are on the same line, hence represent the same point of $\mathbb{C}H^2$. Therefore $\boldsymbol{x}$ represents a fixed point of A. □

Lemma 4. *For $\Gamma_\theta \colon \triangle_p \to SU(2,1)$, the eigenvalue associated with the fixed point (i.e. triangle vertex) of each generator a , b , c is* 1.

Proof. Let i_1, i_2, and i_3 be reflections in Lagrangian planes such that $a = i_1 i_2$, $b = i_2 i_3$, and $c = i_3 i_1$ in accordance with the setup of Chapter 3. Recall that in Chapter 3, when constructing the Lagrangian planes P_1, P_2, P_3, we began by considering the plane $Q = \{[x , ye^{i\theta} , 1]^T \mid x, y \in \mathbb{R}\}$ containing the x–axis γ and inclined at angle θ to P_0, the canonically embedded $\mathbb{R}H^2$. We recall the notation α for the angle at each vertex of our fixed equilateral triangle, and t , r for the translation and rotation used for mapping γ to the sides of the triangle.

Let s_α denote rotation through angle α in $\mathbb{R}H^2$ about the origin. From the construction of Chapter 3, the isometry t maps Q to P_1; since the angle between the triangle sides $P_1 \cap P_0$ and $P_2 \cap P_0$ is α, the translation t maps $s_\alpha(Q)$ to P_2. It follows that the generator $a = i_1\, i_2$ is conjugate via t^{-1} to the product of inversions in Q , $s_\alpha(Q)$. Similarly, the other generators b , c are also conjugate to this product. Therefore it is sufficient to check that the conclusion holds for the product of inversions in Q , $s_\alpha(Q)$, and this is the matrix

$$A = \begin{bmatrix} \cos^2\alpha + e^{-2i\theta}\sin^2\alpha & \sin\alpha\cos\alpha\left(1 - e^{-2i\theta}\right) & 0 \\ \left(e^{2i\theta} - 1\right)\sin\alpha\cos\alpha & e^{2i\theta}\sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

The fixed point of A is of course the origin $[0,0,1]^T$, and it is immediate from inspection of the last column of A that the associated eigenvalue is 1, QED. □

4.2 Normalized Representations

The construction described above gives a practical method for computing numerical approximations for the generating matrices a , b , c for given p , θ, but exact versions of these matrices are far too cumbersome at present. A dramatic improvement is obtained by adopting a different basis for $\mathbb{C}^{2,1}$, namely a basis consisting of vectors representing the fixed points of a , b , c.

Let A, B, C be matrices in $SU(2,1)$ representing the triangle group generators a, b, c respectively. Let $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ be eigenvectors of A, B, C, each with eigenvalue 1, and let P be the matrix whose columns are $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$, written in column form. Then the matrices $A' = P^{-1}AP$, $B' = P^{-1}BP$, $C' = P^{-1}CP$ have the form

$$A' = \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} , \quad B' = \begin{bmatrix} * & 0 & * \\ * & 1 & * \\ * & 0 & * \end{bmatrix} , \quad C' = \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{bmatrix} .$$

Since eigenvectors are only determined up to non-zero scalar multiples, our matrices A' , B' , C' are not yet fully determined. We note that the pattern of 0's and 1's of the matrices A' , B' , C' are related by cyclic permutations of rows and columns. From the geometric symmetry of the configuration, it is then reasonable to suppose that if

$\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ are chosen suitably, then *all entries* of these matrices will be related by cyclic permutations of rows and columns. Specifically, if we write $A' = (a_{i,j})$, $B' = (b_{i,j})$, $C' = (c_{i,j})$, then we hope that by suitable choice of $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ we have $b_{i,j} = a_{i-1,j-1}$ and $c_{i,j} = b_{i-1,j-1}$, where subscripts are taken modulo 3. Not surprisingly, it turns out that this is achieved if we choose $\mathbf{v}_A$, $\mathbf{v}_B$, $\mathbf{v}_C$ all to have the same length.

At this stage, the conjecture of the previous paragraph was verified numerically, for various choices of p and θ. It was hoped that in addition to there being this nice relationship between entries of the three matrices, the matrix entries themselves would admit simple expressions. The matrices are presented in the next chapter; we conclude this chapter with a brief description of how these simple expressions were guessed. As most readers might not be familiar with LLL and its related facility *lindep()*, we also offer a short description of them.

The LLL algorithm [LLL82] takes as input a complex number t in floating-point form, often to several hundred decimal places, a maximum degree $d \geq 1$ and a working precision k. In return, one receives a polynomial in $\mathbb{Z}[x]$ of degree at most d, of which t is a root to within the working precision. For example, the input $t = 1.41421356$, $d = 2$, $k = 6$ yields the answer $x^2 - 2$.

Of course, the input t is always rational if interpreted literally, and one should be suspicious of any polynomial whose coefficients are large relative to the working precision k. So one should reject

$$t = 3.1415926535\ ,\ d = 10\ ,\ k = 10 \quad \longrightarrow \quad 3x^{10}-9x^9-5x^7+9x^5-5x^4+3x^3+6x^2+6x+1\ .$$

Using *Mathematica*, numerical approximations were computed to 500 decimal places for the matrices A' , B' , C', for various small values of p, and various values of θ for which $\cos\theta$ was a rational number. LLL was applied to the matrix entries, and the degrees of the resulting polynomials strongly suggested that the matrix entries might be in a quadratic extension of the field $\mathbb{Q}\left(e^{2i\theta}\,,\,cos\frac{2\pi}{p}\right)$. Indeed, general expressions for some of the matrix entries could be guessed at this point.

Encouraged by these findings, the next step was to use a related facility to guess radical expressions for the remaining matrix entries in terms of $e^{2i\theta}\,,\,cos\frac{2\pi}{p}$. In the number theory package *gp* [BBB$^+$90], this facility is implemented as the function *lindep().* This function guesses integer linear relations between the components of a vector, where again the vector is input in floating-point form. For example, if we set t to be an apporximation of $2^{1/2}+3^{1/3}$ to 150 decimal places, the command

$$\text{lindep}\,(\,[1\;,\;t\;,\;t^2\;,\;t^3\;,\;t^4\;,\;t^5\;,\;2^{1/2}]\;,\;100)$$

yields the answer

$$[-1092\;,\;879\;,\;-468\;,\;-320\;,\;27\;,\;48\;,\;-755]\quad,$$

from which one deduces that

$$\sqrt{2}=\frac{1}{755}\left(-1092+879t-468t^2-320t^3+27t^4+48t^5\right)\quad.$$

The function lindep() was successfully applied in this manner to the entries of A', yielding the formulae given in the statement of the Main Theorem of Chapter 5. Of course, so far these expressions are the result of guesswork, and they need to be proved.

Using the normalized generating matrices A', B', C' has several potential advantages. First, the representations have geometric significance as the vertices of the base triangle were used as basis vectors for the generators. The possibilities of using such generators has not been fully explored. These representations also reflect the symmetry of the generators, in fact, each generator contains the same entries, differing only by cycling rows and columns. Also, although performing calculations, such as finding traces of group elements, may still be difficult, one would expect calculations using these representations to be greatly simplified.

Chapter 5

The Main Theorem

5.1 The Generators of the Representations

We are now ready to state a theorem which presents the results of the procedure described at the end of the last chapter. Here we present matrices which generate representations of $\triangle_p$ into $SU(2,1)$ which only depend on the angle of inclination of the Lagrangian planes and, of course, p. Again, we note that the entries of each generating matrix differ from the others only by cycling rows and columns. We may also observe that a, b, and c have the relationship $c^{-1} = ab$.

Main Theorem. *Let $4 \leq p \in \mathbb{N}$. The matrices a, b, and c in $SU(2,1)$ with entries described below generate a Langrangian representation of the (p,p,p)-triangle group for each θ such that $\theta_p < \theta \leq \frac{\pi}{2}$ where θ is the angle of inclination of the corresponding Lagrangian planes.*

$$a = \begin{bmatrix} 1 & 2i\eta e^{-i\theta} & -2i\eta e^{i\theta} + 4\eta^2 \\ 0 & e^{-2i\theta} & -2i\eta e^{i\theta} \\ 0 & -2i\eta e^{i\theta} & 2\cos\frac{2\pi}{p} - e^{-2i\theta} \end{bmatrix}$$

$$b = \begin{bmatrix} 2\cos\frac{2\pi}{p} - e^{-2i\theta} & 0 & 2i\eta e^{-i\theta} \\ -2i\eta e^{i\theta} + 4\eta^2 & 1 & -2i\eta e^{i\theta} \\ -2i\eta e^{i\theta} & 0 & e^{-2i\theta} \end{bmatrix}$$

$$c = \begin{bmatrix} e^{-2i\theta} & -2i\eta e^{i\theta} & 0 \\ -2i\eta e^{i\theta} & 2\cos\frac{2\pi}{p} - e^{-2i\theta} & 0 \\ 2i\eta e^{-i\theta} & -2i\eta e^{i\theta} + 4\eta^2 & 1 \end{bmatrix},$$

where

$$\eta = \sqrt{\sin\left(\theta + \frac{\pi}{p}\right)\sin\left(\theta - \frac{\pi}{p}\right)} = \sqrt{\frac{1}{2}\left(\cos\frac{2\pi}{p} - \cos 2\theta\right)}.$$

Most of the remainder of this chapter will be dedicated to proving the theorem above. The proof will involve three major sections. First, we will address the properties necessary for each individual generating matrix, that is we will check that each matrix is unitary with respect to a Hermitian form of signature $(2, 1)$ and has determinant one. Then the necessary group relations will be verified. Finally, we prove that the representations are in fact Lagrangian. As earlier noted, each step in this process is absolutely necessary due to the imprecision and guesswork used in developing these generators. Going forward, we will make little distinction between the abstract generators of the group $\triangle_p$ and the matrices that represent these generators.

5.2 Relationships Among Angles

Before we continue toward proving the Main Theorem, we recall and establish some relationships among important angles involved in the deformation of the representations. Recall that for a fixed p, our representations are deformed by decreasing the angle of inclination, θ. As θ changes the angles of the base triangle (in the standard projective model), each measuring α, change as well. Initially when $\theta = \frac{\pi}{2}$, $\alpha = \frac{\pi}{p}$. There are two main relationships we will use occasionally. The first was established in Theorem 4.

$$\sin\alpha\sin\theta = \sin\frac{\pi}{p}$$

The second relationship relates $\cos\alpha$ with θ and is useful for ease in calucation.

Lemma 5. *Let*

$$\eta = \sqrt{\sin\left(\theta + \frac{\pi}{p}\right)\sin\left(\theta - \frac{\pi}{p}\right)}$$

Then

$$\eta = \sqrt{\cos^2\frac{\pi}{p} - \cos^2\theta} = \sqrt{\sin^2\theta - \sin^2\frac{\pi}{p}} = \cos\alpha\sin\theta$$

Proof. The proof follows from trigonometric identities, Theorem 2, and the fact that $0 \leq \alpha, \theta \leq \frac{\pi}{2}$. □

5.3 Representations and Associated Hermitian Forms

Until this point, we have used the First Hermitian form defined by

$$\langle \boldsymbol{w}, \boldsymbol{z}\rangle_1 = w_1 z_1 + w_2 z_2 - w_3 z_3$$

for calculating inner products. This Hermitian form is used when working with the ball model or the projective model of $\mathbb{C}H^2$. But, as noted earlier, any Hermitian form of signature $(2, 1)$ could be used in some model of $\mathbb{C}H^2$. During the deformation of our representations as seen from the projective model viewpoint of Chapter 3, the angles of the base triangle change as θ changes. Hence it must be true that the vertices of the base triangle change as well. But, throughout the curve of representations given by the main theorem the vertices of the base triangle are constantly given by the coordinates $[1, 0, 0]^T$, $[0, 1, 0]^T$, and $[0, 0, 1]^T$. This phenomenon can be explained in terms of the construction used in Chapter 4. When each generator is conjugated by a matrix of eigenvectors, we essentially change the basis of our matrices to the vectors which represent the vertices of the base triangle. Thus, we observe that in the main theorem the basis for the matrices changes along with the vertices. Two consequences are that the First Hermitian form is no longer valid and that a different Hermitian form is needed for each θ along the curve of representations. We are essentially using a different model of $\mathbb{C}H^2$ for each value of θ. The next two lemmas explicitly define and validate the Hermitian forms used throughout the deformation.

Lemma 6. *Let*

$$\kappa = 1 - \frac{\sin\theta}{\eta}$$

Then for every θ such that $\theta_p < \theta \leq \frac{\pi}{2}$, $\langle \quad , \quad \rangle_\theta$ defined by

$$\langle \boldsymbol{z}, \boldsymbol{w} \rangle_\theta = \kappa\left(z_1\overline{w_1} + z_2\overline{w_2} + z_3\overline{w_3}\right) - \left(z_1\overline{w_2} + z_2\overline{w_3} + z_3\overline{w_1} + z_1\overline{w_3} + z_2\overline{w_1} + z_3\overline{w_2}\right)$$

is a Hermitian form of signature $(2,1)$.

Proof. $\langle \boldsymbol{z}, \boldsymbol{w} \rangle_\theta$ is defined using the matrix

$$J_\theta = \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix}$$

where $\langle \boldsymbol{z}, \boldsymbol{w} \rangle_\theta = \boldsymbol{w}^* J_\theta \boldsymbol{z}$ and $*$ indicates the conjugate transpose. Since J_θ is symmetric with real entries, J_θ is Hermitian. We now only have to check that J_θ has two positive eigenvalues and one negative eigenvalue. The characteristic polynomial for J_θ is

$$\begin{aligned} &(\lambda - \kappa)^3 - 1 - 1 - ((\lambda - \kappa) + (\lambda - \kappa) + (\lambda - \kappa)) \\ &= \lambda^3 - 3\lambda^2\kappa + 3\lambda\kappa^2 - 3\lambda - \kappa^3 + 3\kappa - 2 \\ &= (\lambda - \kappa + 2)(\lambda - \kappa - 1)^2 \end{aligned}$$

So J_θ has eigenvalue $\lambda = \kappa - 2$ and repeated eigenvalue $\lambda = \kappa + 1$. One may readily observe that $\kappa - 2$ is negative for all θ such that $\theta_p < \theta \leq \frac{\pi}{2}$. Observe that

$$\kappa + 1 = 2 - \frac{\sin\theta}{\eta} = 2 - \frac{1}{\cos\alpha}$$

Since $0 < \alpha < \frac{\pi}{3}$, we know $\frac{1}{2} < \cos\alpha < 1$. Hence, $0 < \kappa + 1 < 1$ for every θ such that $\theta_p < \theta \leq \frac{\pi}{2}$. ☐

Now, we need to check that J_θ induces the correct Hermitian forms.

Lemma 7. *For every θ such that $\theta_p < \theta \leq \frac{\pi}{2}$, a, b, and c are unitary with respect to $J = J_\theta$.*

Proof. A straightforward, but lengthy calculation shows $a^* J a = J$, $b^* J b = J$, and $c^* J c = J$. ☐

Next, we check the determinants of the generators.

Lemma 8. $\det(a) = \det(b) = \det(c) = 1$

Proof. For a, b, and c using cofactor expansion along the column containing 1 yields

$$\begin{aligned}
&e^{-2i\theta}\left(2\cos\frac{2\pi}{p} - e^{-2i\theta}\right) - \left(2i\eta e^{-i\theta}\right)\left(-2i\eta e^{-i\theta}\right) \\
&= e^{-2i\theta}\left(-4\cos^2\theta - 2 - \cos 2\theta + i\sin 2\theta\right) \\
&= e^{-2i\theta}\left(2\cos 2\theta - \cos 2\theta + i\sin 2\theta\right) \\
&= e^{-2i\theta}\left(\cos 2\theta + i\sin 2\theta\right) = e^{-2i\theta}e^{2i\theta} = 1
\end{aligned}$$

☐

5.4 Generators and Group Relations

In this section we continue the proof of the main theorem by verifying that the necessary relations among generators.

Lemma 9. *The generators a, b, and c satisfy the group relations $a^p = b^p = c^p = 1$.*

Proof. a, b, and c all have characteristic polynomial

$$
\begin{aligned}
&(\lambda - 1)\left(\left(\lambda - e^{-2i\theta}\right)\left(\lambda - 2\cos\frac{2\pi}{p} + e^{-2i\theta}\right) - \left(2i\eta e^{-i\theta}\right)\left(-2i\eta e^{-i\theta}\right)\right) \\
&= (\lambda - 1)\left(\lambda^2 - 2\lambda\cos\frac{2\pi}{p}\right) + 2e^{-2i\theta}\cos\frac{2\pi}{p} - e^{-4i\theta} - 4\eta^2 e^{-2i\theta} \\
&= (\lambda - 1)\left(\lambda^2 - 2\lambda\cos\frac{2\pi}{p} + 1\right) \\
&= (\lambda - 1)\left(\lambda^2 - \lambda\left(e^{\frac{2\pi i}{p}} + e^{-\frac{2\pi i}{p}}\right) + 1\right) \\
&= (\lambda - 1)\left(\lambda - e^{\frac{2\pi i}{p}}\right)\left(\lambda - e^{-\frac{2\pi i}{p}}\right)
\end{aligned}
$$

Thus a, b, and c have eigenvalues 1, $e^{\frac{2\pi i}{p}}$, and $e^{-\frac{2\pi i}{p}}$. The eigenvalues are distinct, so each generator is diagonalizable and can written in the form PQP^{-1} for some matrix P where

$$
Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{p}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{p}} \end{bmatrix}
$$

$Q^p = I$ since each of the eigenvalues are p-th roots of unity. Thus,

$$
\left(P^{-1}QP\right)^p = P^{-1}Q^pP = P^{-1}IP = P^{-1}P = I
$$

Hence, each generator has order p. □

Lemma 10. *The generators a, b, and c satisfy the group relation $abc = 1$.*

For the sake of completeness, a detailed proof of Lemma 10 is given in the appendix.

5.5 Lagrangian Criteria

The only task that remains in proving the main theorem is to show that the representations are Lagrangian. Recall that a representation is Lagrangian if the vertices of the base triangle lie in a Lagrangian plane. In order to prove the representations of the main theorem are Lagrangian, we will use the following lemma.

Lemma 11. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{C}^{2,1}$, the points in $\mathbb{C}H^2$ represented by $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ lie in a Lagrangian plane if $\langle \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \rangle \in \mathbb{R}, \setminus \{0\}$.*

This lemma is a corollary of Lemma 2.2.5 of [Gol99].

Theorem 5. *The representations of $\triangle_p$ generated by a, b, and c are Lagrangian.*

Proof. Let $\boldsymbol{z}_1 = [1,0,0]^T$, $\boldsymbol{z}_2 = [0,1,0]^T$, and $\boldsymbol{z}_3 = [0,0,1]^T$. Note that

$$\langle \boldsymbol{z}_1, \boldsymbol{z}_1 \rangle_\theta = [1,0,0] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \kappa$$

$$\langle \boldsymbol{z}_2, \boldsymbol{z}_2 \rangle_\theta = [0,1,0] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \kappa$$

$$\langle \boldsymbol{z}_3, \boldsymbol{z}_3 \rangle_\theta = [0,0,1] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \kappa$$

It was shown in the proof of Lemma 7 that $0 < \kappa + 1 < 1$. So, $-1 < \kappa < 0$. Hence, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$ represent points in $\mathbb{C}H^2$ for all $\theta_p < \theta \leq \frac{\pi}{2}$.

$$\langle \boldsymbol{z}_1, \boldsymbol{z}_2 \rangle_\theta = [0,1,0] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -1$$

$$\langle \boldsymbol{z}_2, \boldsymbol{z}_3 \rangle_\theta = [0,0,1] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -1$$

$$\langle \boldsymbol{z}_3, \boldsymbol{z}_1 \rangle_\theta = [1,0,0] \begin{bmatrix} \kappa & -1 & -1 \\ -1 & \kappa & -1 \\ -1 & -1 & \kappa \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1$$

Thus, the Hermitian triple product

$$\langle \boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3 \rangle_\theta = \langle \boldsymbol{z}_1, \boldsymbol{z}_2 \rangle_\theta \langle \boldsymbol{z}_2, \boldsymbol{z}_3 \rangle_\theta \langle \boldsymbol{z}_3, \boldsymbol{z}_1 \rangle_\theta = -1 \in \mathbb{R}$$

Hence, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$ lie in a Lagrangian plane, by the previous lemma. One may quickly observe that a, b, and c have fixed points $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, and $\boldsymbol{z}_3$, respectively. So, a, b, and c may each be written as a product of two reflections in Lagrangian planes by Theorem 1. □

Fig. 4 illustrates the space of representations of Δ_4 into $SU(2,1)$. The coordinates are essentially the real and imaginary parts of $3\text{tr}(a\,b^{-1})$, although to ensure that the singularity is at the origin the figure is translated horizontally by -2 units. The "base representation" into isometries of $\mathbb{R}H^2$ is located on the horizontal axis at the "nose" of the fish. Our Lagrangian represention follow the boundary of the fish from the nose up to the singular point at the origin, either above or below the horizontal axis. The representations situated on the horizontal axis have been investigated by numerous authors, in particular [Pra05]. The representations in the tail of the fish, to the left of the origin, are in fact not into $SU(2,1)$ but into the unitary group $SU(3)$, and correspond to isometries of *elliptic space*, endowed with the *Fubini-Study metric* [Gol99]. It is very likely that a slight adaptation of our matrices would provide generators for representations on the boundary of the tail, but this has yet to be investigated.

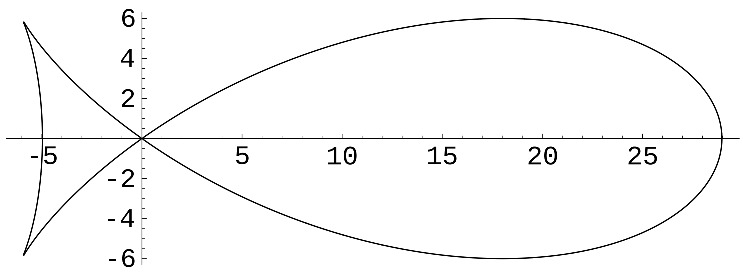

Fig.4: The space of representations of Δ_4 *into* $SU(2,1)$

Chapter 6

Parabolicity of Words in $\triangle_p$

A fundamental problem is to decide whether a given group of isometries of $\mathbb{C}H^2$ is discrete. It is known ([CLT07], Theorem 2.3) that discreteness holds for θ in some neighborhood of $\frac{\pi}{2}$, but establishing discreteness for any particular $\theta \neq \frac{\pi}{2}$ is difficult, involving the construction of a fundamental domain for the action of the group. In the complex hyperbolic plane, there are serious technical difficulties in constructing these fundamental domains, as evidenced by errors in published papers. However, it is much easier to find θ_0 such that, given $\epsilon > 0$, discreteness does *not* hold for all $\theta \in (\theta_0 - \epsilon\ ,\ \frac{\pi}{2})$. Specifically, if $g \in \triangle_p$ is regular elliptic and of infinite order, then the orbit of any point not fixed by g will have an accumulation point, implying that the action of the subgroup generated by g is not discrete. This situation is analogous to that of a Euclidean rotation of infinite order about a point in the plane.

In this chapter we examine the behavior of the elements

$$g_1 = [a\,,\,b] = aba^{-1}b^{-1}\ \ ,\ \ g_2 = aabb\ \ ,\ \ g_3 = abab^{-1}\ \ ,\ \ g_4 = aaabbb$$

in this regard. We shall show by direct computation that there exists θ_i $(i = 1, 2, 3, 4)$ such that g_i is hyperbolic for $\theta \in (\theta_i\ ,\ \frac{\pi}{2})$, g_i is parabolic for $\theta = \theta_i$ and g_i is regular elliptic within an interval $(\theta_i - \epsilon\ ,\ \theta_i)$. Since for generic θ each of g_1 , g_2 , g_3 , g_4 has infinite order, it will follow that the action of the triangle group cannot be discrete within the interval $(\theta_i - \epsilon\ ,\ \theta_i)$, except possibly at isolated points.

We note that we cannot establish the exact interval of discreteness by this method, as there are infinitely many group elements to check, and it is conceivable that discreteness could fail for some reason other than ellipticity of an element.

From [Gol99], an isometry is hyperbolic (respectively parabolic, elliptic) if and only if its trace τ lies outside (respectively on, inside) the deltoid

$$f(\tau) \;\; := \;\; |\tau|^4 - 8\Re(\tau^3) + 18|\tau|^2 - 27 \;\; = \;\; 0 \;\; .$$

For each of the elements g_1 , g_2 , g_3 , g_4 under consideration, the situation is simplified by the facts $tr(g_i)$ is real and $tr(g_i)$ decreases as θ decreases. For real τ the expression $f(\tau)$ factors as $(\tau - 3)^3(\tau + 1)$, so we see that g_i first becomes parabolic when $tr(g_i) = 3$.

The verification that $tr(g_i)$ decreases as θ decreases relies on an elementary but tedious calculation with its derivative with respect to θ, and is omitted. It would be very interesting to find a simple criterion for deciding which words in a , b have real trace. For example, the trace of ab^{-1} is not real, and finding the exact value of θ where ab^{-1} becomes parabolic is harder than for the words g_i dealt with here.

The following computations are elementary, but as they are somewhat complicated we rely on *Mathematica*. Here all computations are exact, and *Mathematica* is without doubt more trustworthy in performing them than a human being.

$\boxed{g_1 = [a\,,\,b] = aba^{-1}b^{-1}}$ We have

$$tr\,(g_1) = 9 + 2\cos\frac{6\pi}{p} + 6\cos\frac{4\pi}{p}\,(1 - 4\cos 2\theta) - 36\cos 2\theta$$
$$+ 4\sqrt{2}\sqrt{\cos\frac{2\pi}{p} - \cos 2\theta}\left(7 + 3\cos\frac{4\pi}{p} + 4\cos 2\theta - 2\cos\frac{2\pi}{p}\,(2 + 7\cos 2\theta) + 4\cos 4\theta\right)\sin\theta$$
$$+ 6\cos\frac{2\pi}{p}\,(5 - 2\cos 2\theta + 6\cos 4\theta) - 8\cos 6\theta$$

For notational convenience, in what follows let u denote $\cos^2\frac{\pi}{p}$. With some help, *Mathematica* obtained the solutions $\theta = \frac{\pi}{p}$ and

$$\cos^2\theta = \frac{16u^2 - 40u + 9}{16u - 40}$$

to the equation $tr(g_1) = 3$. It is easily verified that the expression given in the second solution takes the value $\frac{7}{32}$ for $p = 4$ and increases to the limit $\frac{5}{8}$ as $p \to \infty$. Since $\cos^2(\pi/5) \approx 0.65$ is already greater than $\frac{5}{8}$, the value of θ given by the second solution is closer to $\frac{\pi}{2}$ than $\theta = \pi/p$, for all $p \geq 4$. Therefore we take

$$\theta_1 = \arccos\sqrt{\frac{16u^2 - 40u + 9}{16u - 40}}\quad.$$

$\boxed{g_2 = aabb}$ This time we have

$$tr\,(g_2) = \left(1 + 2\cos\frac{2\pi}{p}\right)\left(5 + 2\cos\frac{4\pi}{p} + 6\,(1 - 2\cos 2\theta)\cos\frac{2\pi}{p} - 4\cos 2\theta\right.$$
$$\left. + 4\cos 4\theta + 8\sqrt{2}\sin\theta\left(\cos\frac{2\pi}{p} - \cos 2\theta\right)^{\frac{3}{2}}\right)$$

and *Mathematica* gives the solution

$$\cos^2\theta = \frac{2u - 10u^2 + 12u^3 - 16u^4 + 2\sqrt{-4u^5 + 5u^4 - u^3}}{1 - 5u + 8u^2 - 16u^3}$$

yielding

$$\theta_2 \;=\; \arccos\sqrt{\frac{2u - 10u^2 + 12u^3 - 16u^4 + 2\sqrt{-4u^5 + 5u^4 - u^3}}{1 - 5u + 8u^2 - 16u^3}} \quad .$$

We note that $\cos^2\theta_2$ takes the value $\frac{1}{3}$ at $p = 4$, and increases to the limit 1 as $p \to \infty$.

$\boxed{g_3 = abab^{-1}}$ It turns out that $tr\,(g_3) = tr\,(g_2)$, so we have $\theta_3 = \theta_2$.

$\boxed{g_4 = aaabbb}$ The expression for the trace is a little more complicated than that for $tr\,(g_2)$, and *Mathematica* finds that

$$\theta_4 \;=\; \arccos\sqrt{\frac{2 + 18u - 92u^2 + 128u^3 - 64u^4 + \sqrt{-2 + 6u - 4u^2}}{8(3 - 12u + 16u^2 - 8u^3)}} \quad .$$

The words $aab^{-1}ab$, $aabab^{-1}$ also have real trace, but the equation in θ for these words is of degree 3, giving solutions that are very awkward to deal with. We do not know of any elegant way of handling words whose trace is non-real, but approximations to solutions can be found using the bisection method.

The question remains as to which of $\theta_1, \ldots, \theta_4$ is greatest, for given p. We observe that for $p = 6$, $\cos^2\theta_2$ is already greater than $\lim_{p\to\infty}\cos^2\theta_1 = \frac{5}{8}$, and that the value of $\cos^2\theta_2$ is greater than that of $\cos^2\theta_1$ for $p = 4, 5$. Therefore $\theta_2 = \theta_3$ becomes redundant. Similarly, θ_4 is redundant. From this information we have

Theorem 6. *For given p, let* $\alpha_p = \inf\{\beta : \Gamma_\theta$ *is discrete in the interval* $(\beta, \pi/2)\}$. *Then*

$$\alpha_p \;\geq\; \theta_1 \;=\; \arccos\sqrt{\frac{16\cos^4\frac{\pi}{p} - 40\cos^2\frac{\pi}{p} + 9}{16\cos^2\frac{\pi}{p} - 40}} \quad .$$

Finally, we conjecture that the inequality in the statement of Theorem 6 may be replaced by equality, although much work needs to be done to confirm this.

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Appendix

Appendix

Proof of Lemma 10

Proof. It suffices to show that each entry in the matrix abc is equal to the appropriate entry in the identity matrix. Verification of this fact is made easier by making some simple observations. Note that each matrix can be written in terms of the entries of a_{12}, a_{13}, a_{22}, and a_{33} only, so each entry of abc may be written in terms of these entries, also. This follows from the fact that a, b, and c all have the same entries and $a_{12} = a_{32} = -a_{23}$. That $(abc)_{23} = 0 = I_{23}$ is a direct result of the above and does not depend on the expressions involving the angle of inclination for each entry . Next we calculate $(abc)_{13}$. Note that

$$(abc)_{13} = a_{12} + a_{12}^2 + a_{13}a_{22}$$

The calculation is split into two pieces. First, we calculate $a_{12} + a_{13}a_{22}$:

$$
\begin{aligned}
& a_{12} + a_{13}a_{22} = 2ie^{-i\theta}\eta + \left(-2ie^{i\theta}\eta\right)e^{-2i\theta} + 4\eta^2 e^{-2i\theta} \\
& = 2e^{-2i\theta}\left(e^{2i\theta}ie^{-i\theta}\eta - ie^{i\theta}\eta + 2\eta^2\right) \\
& = 2e^{-2i\theta}\left(ie^{i\theta}\eta - ie^{i\theta}\eta + 2\eta^2\right) \\
& = 2e^{-2i\theta}\left(2\sin^2\theta\cos^2\frac{\pi}{p} - 2\sin^2\frac{\pi}{p}\cos^2\theta\right) \\
& = 2e^{-2i\theta}\left(2\sin^2\theta\cos^2\frac{\pi}{p} - 2\left(1 - \cos^2\frac{\pi}{p}\right)\cos^2\theta\right) \\
& = 2e^{-2i\theta}\left(2\sin^2\theta\cos^2\frac{\pi}{p} + 2\cos^2\theta\cos^2\frac{\pi}{p} - 2\cos^2\theta\right) \\
& = 2e^{-2i\theta}\left(2cos^2\frac{\pi}{p} - 2\cos^2\theta\right) \\
& = 2e^{-2i\theta}\left(2\left(\frac{1+\cos\frac{2\pi}{p}}{2}\right) - 2\left(\frac{1+\cos 2\theta}{2}\right)\right) \\
& = 2e^{-2i\theta}\left(\cos\frac{2\pi}{p} - \cos 2\theta\right)
\end{aligned}
$$

Now,

$$
\begin{aligned}
a_{12}^2 &= \left(2ie^{-i\theta}\sqrt{\sin\left(\theta+\frac{\pi}{p}\right)\sin\left(\theta-\frac{\pi}{p}\right)}\right)^2 \\
&= -4e^{-2i\theta}\sin\left(\theta+\frac{\pi}{p}\right)\sin\left(\theta-\frac{\pi}{p}\right) \\
&= -4e^{-2i\theta}\left(\sin^2\theta\cos^2\frac{\pi}{p}-\sin^2\frac{\pi}{p}\cos^2\theta\right) \\
&= -4e^{-2i\theta}\left(\sin^2\theta\cos^2\frac{\pi}{p}+\cos^2\frac{\pi}{p}\cos^2\theta-\cos^2\theta\right) \\
&= -4e^{-2i\theta}\left(\cos^2\frac{\pi}{p}-\cos^2\theta\right) \\
&= -2e^{-2i\theta}\left(2\left(\frac{1+\cos\frac{2\pi}{p}}{2}\right)-2\left(\frac{1+\cos 2\theta}{2}\right)\right) \\
&= -2e^{-2i\theta}\left(\cos\frac{2\pi}{p}-\cos 2\theta\right)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(abc)_{13} &= a_{12}+a_{12}^2+a_{13}a_{22} \\
&= 2e^{-2i\theta}\left(\cos\frac{2\pi}{p}-\cos 2\theta\right)-2e^{-2i\theta}\left(\cos\frac{2\pi}{p}-\cos 2\theta\right) \\
&= 0 = I_{13}
\end{aligned}
$$

Four other entries are multiples of $(abc)_{13}$.

$$\begin{aligned}
(abc)_{12} &= a_{12}a_{13} + a_{12}^2 a_{13} + a_{13}^2 a_{22} \\
&= a_{13}\left(a_{12} + a_{12}^2 + a_{13}a_{22}\right) \\
&= a_{13} \cdot 0 = 0 = I_{12} \\
(abc)_{21} &= a_{12}a_{22} + a_{12}^2 a_{22} + a_{13}a_{22}^2 \\
&= a_{22}\left(a_{12} + a_{12}^2 + a_{13}a_{22}\right) \\
&= a_{22} \cdot 0 = 0 = I_{21} \\
(abc)_{31} &= a_{12}^2 + a_{12}^3 + a_{12}a_{13}a_{22} \\
&= a_{12}\left(a_{12} + a_{12}^2 + a_{13}a_{22}\right) \\
&= a_{12} \cdot 0 = 0 = I_{31} \\
(abc)_{32} &= a_{12}a_{33} + a_{12}^2 a_{33} + a_{13}a_{22}a_{33} \\
&= a_{33}\left(a_{12} + a_{12}^2 + a_{13}a_{22}\right) \\
&= a_{33} \cdot 0 = 0 = I_{32}
\end{aligned}$$

We now examine the entries of abc that lie on the diagonal.

$$\begin{aligned}
&(abc)_{33} = a_{12}^2 + a_{22}a_{33} \\
&= \left(-2e^{-2i\theta}\left(\cos\frac{2\pi}{p} - \cos 2\theta\right)\right)^2 + e^{-2i\theta}\left(2\cos\frac{2\pi}{p} - e^{-2i\theta}\right) \\
&= e^{-2i\theta}\left(-4\cos^2\frac{\pi}{p} + 4\cos^2\theta\right) + e^{-2i\theta}\left(2\cos\frac{2\pi}{p} - e^{-2i\theta}\right) \\
&= e^{-2i\theta}\left(-2\left(1 + \cos\frac{2\pi}{p}\right) + 2\left(1 + \cos 2\theta\right) + 2\cos\frac{2\pi}{p} - \cos 2\theta + i\sin 2\theta\right) \\
&= e^{-2i\theta}\left(\cos 2\theta + i\sin 2\theta\right) = e^{-2i\theta}e^{2i\theta} = 1 = I_{33}
\end{aligned}$$

The other entries on the diagonal follow.

$$
\begin{aligned}
(abc)_{22} &= -a_{12}^3 - a_{12}a_{13}a_{22} + a_{22}a_{33} \\
&= \left(a_{12}^2 + a_{22}a_{33}\right) - \left(a_{12}^2 + a_{12}^3 + a_{12}a_{13}a_{22}\right) \\
&= (abc)_{33} - (abc)_{31} = 1 - 0 = 1 = I_{22}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(abc)_{11} &= 2a_{12}^2 + a_{12}^3 + a_{12}a_{13}a_{22} + a_{22}a_{33} \\
&= \left(a_{12}^2 + a_{22}a_{33}\right) + \left(a_{12}^2 + a_{12}^3 + a_{12}a_{13}a_{22}\right) \\
&= (abc)_{33} + (abc)_{31} = 1 + 0 = 1 = I_{11}
\end{aligned}
$$

□

Vita

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