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A Compilation of Undergraduate Research

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Bachelor of Science

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1 Preface

This work contains research that I have performed during my tenure as an undergraduate at the University of Tennessee. It contains two unrelated projects: "'On Collections of Sets of Prescribed Cardinality with Pairwise Intersection of Prescribed Cardinality"' and "'Redheffer's Matrix and the Riemann Hypothesis."' The first work, combinatorial in nature, is a presentation of original results regarding the maximum size of a collection of n-element subsets of $\{1, ..., k\}$ any two of which intersect in exactly j elements. The latter work, produded in the Research Experience for Undergraduates programs at Louisiana State University in summer 2004, describes an equivalent formulation of the Riemann Hypothesis, which is widely considered one of the most important open problems in mathematics.

On Collections of Sets of Prescribed Cardinality with Pairwise Intersections of Prescribed Cardinality

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1 Introduction

For integers $0 \leq j < n \leq k$, we will refer to a collection of sets as a $(k,n,j) - $*collection* if its members are n-element subsets of $[k] := \{1, ..., k\}$ such that any two intersect in exactly j elements. Let $N(k,n,j)$ be the maximum cardinality of a $(k,n,j) - $*collection.* Although we discuss the relationship between N and the Hadamard conjecture, our purpose is simply to investigate the behavior of N.

2 Bounding N

Given a sequence $H = (H_1, ..., H_h)$ of subsets of $[k]$, we define the h by k matrix $\Gamma_k(H) := (a_{ij})$ where $a_{ij} = \begin{cases} 1, & \text{if } j \in H_i; \\ -1, & \text{if } j \notin H_i. \end{cases}$ Clearly, Γ_k is a bijection between the set of nonempty sequences of subsets $[k]$ and the set of $(1,-1)$-matrices with k columns.

Now, suppose that we have a $(k,n,j) - $*collection.* We can then arrange the members of the collection into a sequence H. Now, each row of $\Gamma_k(H)$ contains n 1's and $k - n$ -1's. Moreover, given two distinct rows, there are j columns in which both have a 1, $2(n-j)$ columns in which one has a 1 and the other has a -1, and $k - 2(n-j) - j$ columns in which both have a -1. Hence, the dot product of any pair of rows is $k - 4(n-j)$.

Lemma 2.1. *Suppose that there is an h by k matrix $A =$ with entries ± 1 such that the dot product of any pair of distinct distinct rows is $\alpha \neq k$. Then $h \leq k$ or $\alpha = -1$.*

Proof. We note that $AA^T = (a_{ij})$ where $a_{ij} = \begin{cases} k, & \text{if } j = i; \\ \alpha, & \text{if } j \neq i. \end{cases}$ Now let $b_{ij} = a_{1j} - (1 - \delta_{1,i})a_{ij}$ and $c_{ij} = b_{ij} + \delta_{1,j} \sum_{u=2}^{h} b_{iu}$. Now, (c_{ij}) is an upper triangular matrix in which one diagonal entry is $k+(h-1)\alpha$ and $h-1$ entries are $k-\alpha$. Hence, $\det(AA^T) = \det(c_{ij}) = (k+(h-1)\alpha)(k-\alpha)^{h-1}$. If $\det(AA^T) = 0$, then $k = -\alpha(h-1)$. Since this is an equation in integers, with h and k positive, we must have $h-1 \leq k$. Hence, $h \leq k$ or $\alpha = -1$. On the other hand, if $\det(AA^T) \neq 0$, then the map $\mathbf{R}^h \to \mathbf{R}^h : x \mapsto AA^Tx$ is an isomorphism, and hence $\mathbf{R}^k \to \mathbf{R}^h : x \mapsto Ax$ is an epimorphism. Thus, $h \leq k$.

□

Proposition 2.2. *If* $k \neq 4m-1$, *then* $N(k, n, n-m) \leq k$.

$N(4m-1, n, n-m) \leq 4m$.

Proof. Suppose we have a $(k, n, n-m)$ – *collection* of cardinality h. We form a sequence H of length h whose components are the members of this collection. Then $\Gamma_k(H)$ is an h by k matrix with any pair of distinct rows having dot product $k - 4m$. By Lemma 2.1, $h \leq k$ if $k \neq 4m-1$, and $h \leq k+1$ if $k = 4m-1$. □

3 The Hadamard Conjecture

The Hadamard Conjecture is the assertion that, for every positive integer m, there is a $4m$ by $4m$ matrix with entries ± 1 in which any two rows are orthogonal. Such a matrix is called a Hadamard matrix.

Suppose that there is a $4m$ by $4m$ Hadamard matrix H. Then multiplying any column by -1 will not change the dot product of any pair of rows. Hence, we may assume that H has only 1's in its first row. We form a new matrix H' by discarding the first row. Now, any row of H' must contain exactly $2m$ 1's. Now, there must be exactly m rows in which any two rows of H' both have a 1. Hence, $\Gamma_k^{-1}(H')$ is a $(4m, 2m, m)$ – *collection* of size $4m-1$, and $N(4m, 2m, m) \geq 4m-1$. Now, by lemma 2.1, the existence of a $4m$ by $4m$ Hadamard matrix implies that $N(4m, 2m, m) = 4m-1$. Conversely, suppose $N(4m, 2m, m) = 4m-1$. Then there is a $(4m, 2m, m)$ – *collection* of size $4m-1$. We arrange the members of this collection into a $(4m-1)$-tuple H and form $\Gamma_{4m}(H)$. We then add a the row $(1, 1, ..., 1)$ to this matrix. This yields a $4m$ by $4m$ Hadamard matrix. We have shown the following:

Remark 3.1. *The Hadamard Conjecture is true if and only if* $N(4m, 2m, m) = 4m-1$ *for every positive integer* m.

Similarly, we can transform every entry of the first row and first column of H to a 1, and we see the following:

Remark 3.2. *The Hadamard Conjecture is true if and only if* $N(4m-1, 2m-1, m-1) = 4m-1$ *for every positive integer* m.

4 The Behavior of N

We begin with a general result that will allow us to evaluate N at certain points. First, we introduce a set which is necessary for this result.
Let $\Omega_{k,n,m} := \left\{\Psi \subset 2^{2^{[n]}\times 2^{[k]-[n]}} : |A \cap C| + |B \cap D| = m = |A| = |B| \ \forall \ \texttt{distinct} \ (A, B), (C, D) \in \Psi\right\}$.

Proposition 4.1. $N(k, n, n-m) = 1 + \max_{\Psi\in\Omega_{k,n,m}} |\Psi|$.

Proof. Let $H_1, ..., H_h$ be the members of a $(k, n, n-m)-$ *collection*. WLOG, we may assume that $H_1 = [n]$. Since $|H_i| = n$ and $|H_i \cap H_1| = n - m$ for all $i > 1$, it follows that $H_i = ([n]-P_i)\cup Q_i$ for $i > 1$ where $P_i \subset [n]$, $Q_i \subset [k]-[n]$, and $|P_i| = m = |Q_i|$. For $i \neq j$, we have
$n-m = |H_i \cap H_j| = |[n]-(P_i \cup P_j)|+|Q_i \cup Q_j| = n-(2m-|P_i \cap P_j|)+|Q_i \cap Q_j|$, so $|P_i \cap P_j| + |Q_i \cap Q_j| = m$.

Thus, $\{(P_i, Q_i) : i \in [h]-1\} \in \Omega_{k,n,m}$.
Hence, $N(k,n,n-m) - 1 \leq \max_{\Psi\in\Omega_{k,n,m}} |\Psi|$.

Conversely, suppose $\{(P_i, Q_i) : i \in [h]\} \in \Omega_{k,n,m}$. Then $|[n] \cap ([n] - P_i)| = n - m$ and $|([n] - P_i) \cup Q_i| = n$. Additionally, for $i \neq j$, we have $|(([n] - P_i) \cup Q_i) \cap (([n] - P_j) \cup Q_j)| = |[n] - (P_i \cup P_j)| + |Q_i \cap Q_j| = n-(2m-|P_i \cap P_j|)+|Q_i \cap Q_j| = n-m$. Hence, $\{[n]\}\cup\{([n] - P_i) \cup Q_i : i \in [h]\}$ is a $(k, n, n-m)-$ *collection* with $h + 1$ members.
Therefore, $1 + \max_{\Psi\in\Omega_{k,n,m}} |\Psi| \leq N(k, n, n-m)$. □

Corollary 4.2. $N(k,n,n-1) = \begin{cases} n+1, & \text{if } 2k > 2n \geq k > n; \\ k-n+1, & \text{if } 2n < k. \end{cases}$

Proof. Suppose that $\Psi \in \Omega_{k,n,1}$ contains $(\{p_1\}, \{q_1\})$, $(\{p_2\}, \{q_2\})$, and $(\{p_3\}, \{q_3\})$. Now, we must have $p_1 = p_2$ or $q_1 = q_2$. Suppose, WLOG that $p_1 = p_2$. Then $q_1 \neq q_2$. Now, if $p_3 \neq p_1 = p_2$, then we must have $q_3 = q_1 \neq q_2 = q_3$, which is impossible. Hence, given $\Psi \in \Omega_{k,n,1}$, then $\Psi = \{(\{p\}, \{q_i\}) : i \in [h]\}$ for some $p \in [n]$ and distinct $q_1, ..., q_h \in [k] - [n]$ or $\Psi = \{(\{p_i\}, \{q\}) : i \in [h]\}$ for some $q \in [k] - [n]$ and distinct $p_1, ..., p_h \in [n]$. Hence, $|\Psi| \leq \max(n, k-n)$. Now, we note that $\{(\{1\}, \{i\}), i \in [k] - [n]\}$ and $\{(\{i\}, \{n+1\}), i \in [n]\}$ are in $\Omega_{k,n,1}$ and contain $k - n$ and n members, respectively. Therefore, $N(k, n, n-1) = 1 + \max(n, k-n)$. □

Proposition 4.3. *If* $k \geq n+m$, *then* $N(k, n, n-m) = N(k, k-n, k-n-m)$.

Proof. Let $f : [k] \to [k] : i \mapsto \begin{cases} k-n+i, & \text{if } i \leq n; \\ i-n, & \text{if } i > n. \end{cases}$ Clearly, f is a bijection.
Now, if $\{(P_i, Q_i) : i \in [h]\} \in \Omega_{k,n,m}$ then $\{(f(Q_i), f(P_i)) : i \in [h]\} \in \Omega_{k,k-n,m}$. Likewise, if $\{(P_i, Q_i) : i \in [h]\} \in \Omega_{k,k-n,m}$, then $\{(f^{-1}(Q_i), f^{-1}(P_i)) : i \in [h]\} \in \Omega_{k,n,m}$.

Therefore, by proposition 4.1, $N(k, n, n-m) = 1 + \max_{\Psi\in\Omega_{k,n,m}} |\Psi| = 1 + \max_{\Psi\in\Omega_{k,k-n,m}} |\Psi| = N(k, k-n, k-n-m)$ □

Corollary 4.4. $N(n+m,n,n-m) = 1 + \lfloor \frac{n}{m} \rfloor$. *If* $k < n+m$, *then* $N(k,n,n-m) = 1$.

Proof. By proposition 4.3, $N(n+m,n,n-m) = N(n+m,m,0) = \lfloor \frac{n+m}{m} \rfloor = 1 + \lfloor \frac{n}{m} \rfloor$.

If $k < n+m$, then there is no $Q \subset [k]-[n]$ with $|Q| = m$. Hence, $\Omega_{k,n,m} = \emptyset$ and $N(k,n,n-m) = 1$. □

Corollary 4.5. $N(k,n,n-m) \leq N(k,2m,m) + 1$.

Proof. For $\{(P_i,Q_i)\} \in \Omega_{k,n,m}$, we note that $\{P_i \cup Q_i\}$ is a $(k,2m,m)-$*collection*. Hence, $\max_{\Psi \in \Omega_{k,n,m}} |\Psi| \leq N(k,2m,m)$. □

In particular, we can show that there is a $4m$ by $4m$ Hadamard matrix by showing that $N(4m,n,n-m) = 4m$ for some n. The next two results will establish that we can show the existence of infinitely many Hadamard matrices in this way. The following result uses a classical construction that Sylvester developed for constructing Hadamard matrices.

Theorem 4.6. $N(4m_1,n_1,n_1-m_1)N(4m_2,n_2,n_2-m_2) \leq$
$N(16m_1m_2, n_1n_2+(4m_1-n_1)(4m_2-n_2), n_1n_2+(4m_1-n_1)(4m_2-n_2)-4m_1m_2)$.

Proof. Given a $(4m_1,n_1,n_1-m_1)-$*collection* of size g and a $(4m_2,n_2,n_2-m_2)-$*collection* of size h, we arrange the members of the collections to form sequences $G = (G_1,...,G_g)$ and $H = (H_1,...,H_h)$. Now, we form the Kronecker product of $\Gamma_{4m_1}(G) = (a_{ij})$ and $\Gamma_{4m_2}(H) = (b_{ij})$, i.e. the gh by $16m_1m_2$ matrix $\Gamma_{4m_1} \otimes \Gamma_{4m_2}(H) := (C_{ij})_{1\leq i\leq g}^{1\leq j\leq 4m_1}$ where $C_{ij} = a_{ij}\Gamma_{4m_2}(H)$ (Note: C_{ij} is a matrix, and we consider the entries of this matrix to be entries of $\Gamma_{4m_1} \otimes \Gamma_{4m_2}(H)$; the matrix itself is not to be considered an entry).

Since each row of $\Gamma_{4m_1}(G)$ contains exactly n_1 ones and each row of $\Gamma_{4m_2}(H)$ contains exactly n_2 ones, it follows that each row of $\Gamma_{4m_1}(G)\otimes\Gamma_{4m_2}(H)$ contains exactly $n_1n_2+(4m_1-n_1)(4m_2-n_2)$ ones. We note that any two rows of $\Gamma_{4m_1}(G)$ have dot product 0, and the same holds for pairs of rows of $\Gamma_{4m_2}(H)$. Now, given two distinct rows of $\Gamma_{4m_1}(G) \otimes \Gamma_{4m_2}(H)$, their dot product is given by $\sum_{u=1}^{4m_1}\sum_{v=1}^{4m_2}(a_{i_1u}b_{j_1v})(a_{i_2u}b_{j_2v}) = (\sum_{u=1}^{4m_1} a_{i_1u}a_{i_2u})(\sum_{v=1}^{4m_2} b_{j_1v}b_{j_2v}) = 0$ since $i_1 \neq i_2$ or $j_1 \neq j_2$. Recall that two vectors with k entries each, n of which are ones and the remaining $k-n$ are negative ones, must have dot product $k-4m$, where $n-m$ is the number of components in which both vectors contain a one. Hence, given any pair of rows of $\Gamma_{4m_1}(G) \otimes \Gamma_{4m_2}(H)$ there are $n_1n_2 + (4m_1-n_1)(4m_2-n_2) - 4m_1m_2$ columns of $\Gamma_{4m_1}(G) \otimes \Gamma_{4m_2}(H)$ in which both contain a one. Now, $\Gamma_{16m_1m_2}^{-1}(\Gamma_{4m_1}(G) \otimes \Gamma_{4m_2}(H))$ is a $(16m_1m_2, n_1n_2 + (4m_1-n_1)(4m_2-n_2), n_1n_2 + (4m_1-n_1)(4m_2-n_2) - 4m_1m_2) -$ *collection* with gh members. □

Corollary 4.7. $N(4^k, 2\cdot 4^{k-1} + 2^{k-1}, 4^{k-1} + 2^{k-1}) = 4^k$ *and*

$N(4^k, 2\cdot 4^{k-1} - 2^{k-1}, 4^{k-1} - 2^{k-1}) = 4^k$.

Proof. This result is a simple induction. First, we note that $N(4,3,2) = 4$ by corollary 4.2. Now, if $N(4^k, 2\cdot 4^{k-1} + 2^{k-1}, 4^{k-1} + 2^{k-1}) = 4^k$, then $4^{k+1} = N(4,3,2)N(4^k, 2\cdot 4^{k-1}+2^{k-1}, 4^{k-1}+2^{k-1}) \leq (4^{k+1}, 2\cdot 4^k+2^k, 4^k+2^k) \leq 4^{k+1}$ by theorem 4.6 and proposition 2.2. Now, $N(4^k, 2\cdot 4^{k-1} - 2^{k-1}, 4^{k-1} - 2^{k-1}) = 4^k$ by proposition 4.3. □

Although the preceding result does establish the existence of a 4^k by 4^k Hadamard matrix, it is more interesting that it contains information about when $N(k,*,*)$ attains a maximum (especially since Sylvester's construction can yeild 2^k by 2^k Hadamard matrices rather than just 4^k by 4^k). There are (at least) two obvious questions that now present themselves. When does equality hold in theorem 4.6? What is $N(4^k, n, n-4^{k-1})$ for $2\cdot 4^{k-1} < n < 2\cdot 4^{k-1}+2^{k-1}$?

Theorem 4.8. *If p is 1 or a prime, then $N(1+p(p+1), p+1, 1) = 1+p(p+1)$.*

Proof. By corollary 4.2, $N(3,2,1) = 3$. Now, for p prime, we begin with the sets $\{i + jp : i \in [p]\} \cup \{1 + p(p+1)\}$ for $j = 0, 1, ..., p$, and $\{1\} \cup \{j + ip : i = 1, ..., p\}$ for $j = 1, ..., p$. Now, we note that each of these sets contains $p+1$ elements and any pair intersect in exactly 1 element.

Now, the map from the set of words of length $p+1$ in the characters $1, 2, ..., p$ to the set of $p+1$-element sets which interect $\{i + jp : i \in [p]\}$ in exactly one element for $j = 0, 1, ..., p$ given by $\phi_0\phi_1...\phi_p \mapsto \{ip + \phi_i : i = 0, ..., p\}$ is a bijection. To see this, we note that a set belonging to the range must have the form $\{jp + i_j : j = 0, 1, ..., p, \ \texttt{with} \ 1 \leq i_j \leq p\}$. Now
$\{jp + i_j : j = 0, 1, ..., p, \ \texttt{with} \ 1 \leq i_j \leq p\} \mapsto i_0 i_1 ... i_p$ is the inverse of the aforementioned map, showing that it is indeed bijective.

Now, we may identify permutations of $[p]$ with words of length p in which each character $1, ..., p$ appears once via the bijection $\phi \mapsto \phi(1)\phi(2)...\phi(p)$. The set corresponding to the word $\phi_0\phi_1...\phi_p$ intersects $\{1\} \cup \{j + ip : i = 1, ..., p\}$ in exactly one element for each $j = 1, ..., p$ if and only if $\phi_0 \neq 1$ and $\phi_1...\phi_p$ corresponds to a permutation of $[p]$.

There is a primitive root γ modulo p, i.e., γ is an integer such that $\gamma^u \equiv 1 (modp)$ if and only if $p-1|u$. Throughout the remainder of this paragraph, identify each expression in the definition or evaluation of a permutation at a point with the member of $\{1, ..., p\}$ to which it is congruent modulo p. Now, we consider the permutations $\delta := (1 \ \gamma \ \gamma^2 ... \gamma^{p-2})$ and $\sigma := (1 \ 2 ... p)$. We note that $\sigma^{i_1}\delta^{j_1}(t) = \sigma^{i_2}\delta^{j_2}(t)$ if and only if $\sigma^{i_1-i_2}\delta^{j_1-j_2}$ fixes t. We note that $\sigma^i\delta^j(p) = \sigma^i(p) = p + i$, so $\sigma^i\delta^j(p) = p \Leftrightarrow i \equiv 0 (modp)$. Now, let $0 \leq i \leq p-1$, $0 \leq j \leq p-2$ and note that $\sigma^i\delta^j(\gamma^x) = \gamma^{j+x} + i$. But $\gamma^{j+x} + i = \gamma^x \Leftrightarrow \gamma^x(\gamma^j - 1) = -i$. Since the powers of γ generate $\{1, 2, ..., p-1\}$, it follows that $\gamma^x(\gamma^j - 1) = -i$ has a unique solution $x \in \{0, ..., p-2\}$ if $i \neq 0$ and $j \neq 0$, while there is not a solution if $i = 0$ and $j \neq 0$ or if $i \neq 0$ and $j = 0$. Hence, if $0 \leq i_1, i_2 \leq p-1$ and $0 \leq j_1, j_2 \leq p-2$ with $(i_1, j_1) \neq (i_2, j_2)$, then $\sigma^{i_1}\delta^{j_1}(t) = \sigma^{i_2}\delta^{j_2}(t)$ holds for exactly one $t \in [1 + p(p+1)]$ unless $i_1 \neq i_2$ and $j_1 = j_2$, in which case there is no solution t.

Thus, the sets $\{i + jp : i \in [p]\} \cup \{1 + p(p+1)\}$ for $j = 0, 1, ..., p$, and $\{1\} \cup \{j + ip : i = 1, ..., p\}$ for $j = 1, ..., p$, together with the sets corresponding to the words $(j+2)\sigma^i\delta^j(1)...\sigma^i\delta^j(p)$ for $i = 0, ..., p-1$ and $j = 0, ..., p-2$, form a $(1+p(p+1), p+1, 1) -$ *collection* with $1+p(p+1)$ members.

Now, $N(1+p(p+1), p+1, 1) = 1+p(p+1)$ if $1+p(p+1) \neq 4p-1$. We note that $1+p(p+1) = 4p-1 \Leftrightarrow p = 1$ or $p = 2$. But $N(3,2,1) = 3$ as above, and $N(7,3,1) \leq 7$ by the remarks in section 2. $\square$

The preceding result gives us a way to construct large collections of sets in which any pair of sets intersect in exactly one element. Perhaps similar methods will work for the construction of a large collection of sets which intersect in exactly $j > 1$ elements. We can also ask how important primality is for the construction of large collections of sets that intersect in exactly one element.

REDHEFFER'S MATRIX AND THE RIEMANN HYPOTHESIS

NICHOLAS S. BOATMAN

ABSTRACT. In this paper, we will show that the determinant of the Redheffer Matrix is equal to Mertens' function, M(n). From this, it will follow that the Riemann Hypothesis is true if and only if

$$M(n) = O(n^{\frac{1}{2}+\epsilon}).$$

1

First, we define several functions and the Redheffer matrix. The Mobius function is

$$\mu(k) = \begin{cases} 0, & \text{if the square of a prime divides } k \\ (-1)^t, & \text{if } k \text{ is square-free with } t \text{ prime divisors.} \end{cases}$$

Mertens' function is

$$M(x) = \sum_{k \leq x} \mu(k)$$

The n by n Redheffer matrix is

$$A_n = (a_{ij})$$

where

$$a_{ij} = \begin{cases} 1, & \text{if j=1 or } i \mid j \\ 0, & \text{otherwise.} \end{cases}$$

We also find it advantageous to introduce another matrix. Define $R_n(k)$ to be the $(n-1)$ by $(n-1)$ matrix formed by deleting column 1 and row k of A_n. Thus,

$$R_n(k) = (r_{ij})$$

where

$$r_{ij} = \begin{cases} 1, & \text{if } i \mid j+1 \text{ and } i < k \\ 1, & \text{if } i+1 \mid j+1 \text{ and } i \geq k \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1. $\mu(k) = (-1)^{k-1} \cdot \det(R_n(k))$

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Proof. We now form a $(k-1)$ by $(k-1)$ submatrix, $S_n(k)$ using the first $k-1$ rows and columns of $R_n(k)$, noting that $R_n(k)$ is upper triangular everywhere else. We will now perform row and column operations on $S_n(k)$ to make it upper triangular.

We note that $S_n(k) = (s_{ij})$ where

$$s_{ij} = \begin{cases} 1, & \text{if } i \mid j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Now, for each row i such that $s_{i(k-1)} = 0$, we use the row operation

$$\text{row}(m) - \text{row}(i) \mapsto \text{row}(m)$$

for every $m \mid i$, $m < i$. We begin the process with $i = k-2$ and end with $i = 2$.

These row operations yield a new matrix $S'_n(k) = (s'_{ij})$ where

$$s'_{ij} = \begin{cases} 1, & \text{if } i = j+1 \\ 1, & \text{if } j+1 \mid k \text{ and } i \mid j+1 \\ 0, & \text{otherwise.} \end{cases}$$

We will use row and column operations on $S'_n(k)$ as follows. For each proper divisor $N \neq 1$ of k (in order of decreasing number of prime factors), if $s'_{N,k-1} = q \neq 0$, use the column operation

$$\text{column}(k-1) - q \cdot \text{column}(N-1) \mapsto \text{column}(k-1)$$

If $s'_n(k-1) = 0$, use the row operation

$$\text{row}(m) - \text{row}(N) \mapsto \text{row}(m)$$

for each proper divisor m of N.

Now suppose

$$k = \prod_{\alpha=1}^{t} p_\alpha$$

Let N_i the be the set of factors of k consisting of i prime factors. Now, suppose $s'_{N,k-1} = q_i$ for each $N \in N_i$, after all column operations involving the product of more than i primes have been performed. Each $M \in N_j$ divides exactly $\binom{t-j}{i-j}$ members of N_i. So,

$$q_j = 1 - \sum_{i=j+1}^{t-1} \binom{t-j}{i-j} \cdot q_i$$

Now, we will show inductively that

$$q_{t-i} = \begin{cases} 1, & \text{if } i \text{ is odd} \\ -1, & \text{if } i \text{ is even.} \end{cases}$$

We suppose that our claim is true $\forall i = 1, 2, \ldots, j$. Then,

$$q_{t-j-1} = 1 - \sum_{i=t-j}^{t-1} \binom{j+1}{j+1-t+i} \cdot q_i$$

$$= 1 - \sum_{i=1}^{j} \cdot q_{t-j-1+i} = 1 + \binom{j+1}{j} \cdot (-1) + \cdots + \binom{j+1}{1} \cdot (-1)^j$$

Since $\binom{n+m}{n} = \binom{n+m}{m}$, we have

$$q_{t-j-1} = \sum_{i=0}^{j} \binom{j+1}{i} \cdot (-1)^i = (1-1)^{j+1} - (-1)^{j+1} = (-1)^j$$

So, for t odd, we have $q_0 = 1$ and for t even, $q_0 = -1$. After performing $k-2$ row switches, we obtain an upper triangular matrix, and

$$\det(R_n(k)) = \begin{cases} (-1)^{k-2}, & \text{if } t \text{ is odd} \\ (-1)^{k-1}, & \text{if } t \text{ is even.} \end{cases}$$

Thus,

$$(-1)^{k-1} \cdot \det(R_n(k)) = \begin{cases} 1, & \text{if } t \text{ is even} \\ -1, & \text{if } t \text{ is odd.} \end{cases}$$

In the case where k has t distinct prime factors but is divisible by the square of some prime, we consider $S'_n(k)$ and note that

$$\text{row}(1) = \sum_{g=1}^{t} \sum_{\substack{v|k \text{ with} \\ g \text{ prime} \\ \text{factors}}} (-1)^{g-1} \text{row}(v) \text{ , where v is square-free.}$$

Thus, $\det(R_n(k)) = 0$ if k is divisible by the square of a prime. □

Corollary 1.2. $\det(A_n) = M(n)$

Proof. We evaluate $\det(A_n)$ using expansion by minors along the first column, and the result follows immediately from Theorem 1.1. □

Lemma 1.3. *Mobius Inversion Theorem:*

$$\textit{If } \ F(n) = \sum_{d|n} f(d), \quad \textit{then } \ f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$$

Corollary 1.4. *For integers $n > 1$,*

$$\sum_{d|n} \mu(d) = 0$$

Proof. Take the function

$$f(d) = \begin{cases} 1, & \text{if } d = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then we define $F(n) = 1$ so that

$$1 = F(n) = \sum_{d|n} f(d)$$

From the Mobius Inversion Theorem, it follows that

$$f(n) = \sum_{d|n} \mu(d) = 0$$

for $n > 1$.

□

Theorem 1.5.

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \textit{when} \quad \Re(s) > 1$$

.

Proof. Using Corollary 1.4, we see that

$$\sum_{N=1}^{\infty} \frac{1}{N^s} \sum_{d|N} \mu(d) = 1$$

Let $N = mn$. We have

$$\sum_{N=1}^{\infty} \frac{1}{N^s} \sum_{d|N} \mu(d) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)}{(mn)^s} = (\sum_{m=1}^{\infty} \frac{1}{m^s})(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}) = \zeta(s) \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

where $\Re(s) > 1$.

□

Henceforth, all results are given by Titchmarsh in [Ti].

Lemma 1.6. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \ (\sigma > 1)$$

where $a_n = O(\psi(n))$ *for non-decreasing* $\psi(n)$ *and*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^s} = O(\frac{1}{(\sigma - 1)^{\alpha}})$$

as $\sigma \to 1$.

Then if $c > 0$, $\sigma + c > 1$, x *is not an integer, and* N *is the closest integer to* x,

$$\sum_{n<x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w) \frac{x^w}{w} \, dw + O(\frac{x^c}{T(\sigma + c - 1)^{\alpha}}) +$$

$$O(\frac{\psi(2x)x^{1-\sigma}\log x}{T}) + O(\frac{\psi(N)x^{1-\sigma}}{T\,|x-N|})$$

Proof. First, note that $h(w) := \frac{1}{w}(\frac{x}{n})^w$ has only one singularity, occurring at $w = 0$. In fact,

$$h(w) = \sum_{m=-1}^{\infty} (\log^{m+1}\frac{x}{n}) \cdot w^m$$

so that $\mathrm{Res}(h, 0) = 1$.

By the residue theorem,

$$\frac{1}{2\pi i}\int_\gamma h(w)\,dw = 1$$

where γ is a positively oriented, piecewise smooth closed curve containing $w = 0$ in its interior. In particular, we take γ to be the rectangle with vertices at $z - iT$, $z + iT$, $c - iT$, and $c + iT$ with $z < 0$.

Now, we note that $\left|\frac{1}{z \pm iT}(\frac{x}{n})^{z \pm iT}\right| < \frac{1}{\sqrt{z^2+T^2}} < \frac{1}{|z|}$ where $n < x$.

Thus,

$$\left|\int_{z+iT}^{z-iT} h(w)\,dw\right| \leq \int_{z+iT}^{z-iT} |h(w)|\,dw \leq \int_{z+iT}^{z-iT} \frac{1}{|z|} = \frac{2T}{|z|} \to 0 \text{ as } z \to -\infty$$

.

It follows that

$$\frac{1}{2\pi i}(\int_{-\infty-iT}^{c-iT} + \int_{c-iT}^{c+iT} + \int_{c+iT}^{-\infty+iT})h(w)\,dw = 1$$

Now, we use integration by parts and see that

$$\int_{-\infty+iT}^{c+iT} h(w)\,dw = \left[\frac{(\frac{x}{n})^w}{w\log(\frac{x}{n})}\right]_{-\infty+iT}^{c+iT} + \frac{1}{\log(\frac{x}{n})}\int_{-\infty+iT}^{c+iT} \frac{1}{w^2}(\frac{x}{n})^w\,dw$$

But,

$$\left|\int_{-\infty+iT}^{c+iT} \frac{h(w)}{w}\,dw\right| \leq \int_{-\infty+iT}^{c+iT} \left|\frac{h(w)}{w}\right|\,dw \leq \int_{-\infty+iT}^{c+iT} (\frac{x}{n})^{\Re(w)}\frac{1}{|w|^2}\,dw$$

$$< \int_{-\infty}^{c} \left(\frac{x}{n}\right)^c \frac{du}{u^2+T^2} < \left(\frac{x}{n}\right)^c \int_{-\infty}^{\infty} \frac{du}{u^2+T^2}$$

It follows that

$$\int_{-\infty+iT}^{c+iT} h(w)\,dw = O\left(\frac{(\frac{x}{n})^c}{T\log\left(\frac{x}{n}\right)}\right)$$

Likewise for $\int_{-\infty-iT}^{c-iT} h(w)\,dw$.

Therefore,

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} h(w)\,dw = 1 + O\left(\frac{(\frac{x}{n})^c}{T\log\left(\frac{x}{n}\right)}\right) \tag{1}$$

When $n > x$, we use ∞ in place of $-\infty$ and obtain

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} h(w)\,dw = O\left(\frac{\left(\frac{x}{n}\right)^c}{T\log\left(\frac{x}{n}\right)}\right) \tag{2}$$

since h has no singularities in the region in question.

We multiply both sides of equations (1) and (2) by $\frac{a_n}{n^s}$ and sum over all n. This yields

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w}\,dw = \sum_{n<x}\frac{a_n}{n^s} + O\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}\left|\log\left(\frac{x}{n}\right)\right|}\right)$$

If $n < \frac{x}{2}$ or $n > 2x$, then $\left|\log\frac{x}{n}\right| > \log 2$. So,

$$\sum_{\substack{n<\frac{x}{2}\text{ or}\\ n>2x}}\frac{|a_n|}{n^{\sigma+c}\left|\log\frac{x}{n}\right|} = O\left(\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma+c}}\right) = O\left(\frac{1}{(\sigma+c-1)^{\alpha}}\right)$$

If $N < n \le 2x$, take $n = N + r$. It follows that

$$\log\frac{n}{x} \ge \log\frac{N+r}{N+\frac{1}{2}}$$

.

Since $r \ge 1$, $\log\frac{N+r}{N+\frac{1}{2}} \ge \log\left(1+\frac{1}{2N+1}\right)$.

So, there are $A, B > 0$ so that $\log\frac{n}{x} > \frac{Ar}{N} > \frac{Br}{x}$.

Thus,

$$\sum_{N<n\le 2x}\frac{|a_n|}{n^{\sigma+c}\left|\log\frac{x}{n}\right|} < \sum_{N<n\le 2x}\frac{x\,|a_n|}{n^{\sigma+c}Br} = O\left(\psi(2x)x^{1-\sigma-c}\sum_{1\le r\le x}\frac{1}{r}\right)$$

$$= O\left(\psi(2x)x^{1-\sigma-c}\log x\right)$$

Likewise, if $\frac{x}{2} \le n < N$, then

$$\log\frac{x}{n} > \log\frac{x}{N} = \log\left(1+\frac{x-N}{N}\right)$$

So,

$$\sum_{\frac{x}{2}\le n<N}\frac{|a_n|}{n^{\sigma+c}\left|\log\frac{x}{n}\right|} = O\left(\frac{\psi(N)x^{1-\sigma-c}}{|x-N|}\right)$$

Combining these asymptotic formulas for all n, we obtain the desired result.

□

Now, we shall assume the Riemann Hypothesis to be true. With this assumption, the Lindelof Hypothesis follows.

Lemma 1.7. *(Lindelof Hypothesis) We have both*

$$\zeta(\sigma+it)=O(t^{\epsilon})$$

and

$$\frac{1}{\zeta(\sigma+it)}=O(t^{\epsilon})$$
$$\forall\ \epsilon>0\ as\ \ t\to\infty$$

Theorem 1.8. *If the Riemann Hypothesis is true, then*

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$$

converges for all $\sigma>\frac{1}{2}$.

Proof. We will apply Lemma 1.6 with $a_n=\mu(n)$, $f(s)=\frac{1}{\zeta(s)}$, $c=2$, and x half an odd integer. It follows that we can take $\alpha=1$, and $\psi(x)=1$. Since we have assumed the Riemann Hypothesis, we note that

$$\frac{x^w}{w\cdot\zeta(s+w)}$$

has only one singularity in γ, occuring at $w=0$ and its residue is $\frac{1}{\zeta(s)}$.

So, we note that

$$\int_{\gamma}\frac{x^w}{w\cdot\zeta(s+w)}\,dw=\frac{1}{\zeta(s)}$$

where γ is the rectangle with vertices $2-iT$, $2+iT$, $\frac{1}{2}-\sigma+\delta-iT$, $\frac{1}{2}-\sigma+\delta+iT$, where $0<\delta<\sigma-\frac{1}{2}$.

So,

$$\sum_{n<x}\frac{\mu(n)}{n^s}=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^w}{w\cdot\zeta s+w}\,dw+O\left(\frac{x^2}{T}\right)$$

where we eliminated two O-terms because they are each smaller than the one included.

Proceeding,

$$=\frac{1}{2\pi i}\left(\int_{2-iT}^{\frac{1}{2}-\sigma+\delta-iT}+\int_{\frac{1}{2}-\sigma+\delta-iT}^{\frac{1}{2}-\sigma+\delta+iT}+\int_{\frac{1}{2}-\sigma+\delta+iT}^{2+iT}\right)\frac{x^w}{w\cdot\zeta(s+w)}\,dw+\frac{1}{\zeta(s)}+O\left(\frac{x^2}{T}\right)$$

The left and right integrals are each

$$O\left(T^{\epsilon-1}\int_{\frac{1}{2}-\sigma+\delta}^{2}x^u\,du\right)=O\left(T^{\epsilon-1}x^2\right)$$

using Lemma 1.7.

The middle integral is

$$O\left(x^{\frac{1}{2}-\sigma+\delta}\int_{-T}^{T}(1+|t|)^{\epsilon-1}\,dt\right)=O\left(x^{\frac{1}{2}-\sigma+\delta}T^{\epsilon}\right)$$

So,

$$\sum_{n<x}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}+O\left(\frac{x^2}{T}\right)+O\left(T^{\epsilon-1}x^2\right)+O\left(x^{\frac{1}{2}-\sigma+\delta}T^{\epsilon}\right)$$

We take $T=x^3$, and the O-terms become $O\left(x^{-1}\right)$, $O\left(x^{3\epsilon-1}\right)$, and $O\left(x^{\frac{1}{2}-\sigma+\delta+3\epsilon}\right)$.

Since each exponent is negative for ϵ small, each O-term goes to zero as $x\to\infty$.

Thus,

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)}\ \ \forall\ \ \sigma>\frac{1}{2}.$$

Since we have assumed that ζ has no zeroes for $\sigma>\frac{1}{2}$, the sum must converge.

□

Theorem 1.9.

If $\displaystyle\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}$ *converges* $\forall\ \ \sigma>\frac{1}{2}$ *then the Riemann Hypothesis is true.*

Proof. If the sum converges for all $\sigma>\frac{1}{2}$, then it converges uniformly in the same region. Thus, it is analytic. Since it is analytic and is $\frac{1}{\zeta(s)}$ for $\sigma>1$, it must be $\frac{1}{\zeta(s)}$ for $\sigma>\frac{1}{2}$. The Riemann Hypothesis follows immediately.

□

Theorem 1.10. *If the Riemann Hypothesis is true, then* $M(x)=O\left(x^{\frac{1}{2}+\epsilon}\right)$ *for all* $\epsilon>0$.

Proof. We take $a_n=\mu(n)$, $f(s)=\frac{1}{\zeta(s)}$, $c=2$, and x half an odd integer in Lemma 1.6. Now, we take $s=0$ and $\delta>0$. Thus, we have

$$M(x)=\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{x^w}{w\cdot\zeta(w)}\,dw+O\left(\frac{x^2}{T}\right)=O\left(\frac{x^2}{T}\right)+O\left(T^{\epsilon}x^{\frac{1}{2}+\delta}\right)+O\left(x^2T^{1-\epsilon}\right)$$

by logic that is almost identical to Theorem 1.8.

Taking $T=x^2$, we obtain

$$M(x)=O\left(x^{\frac{1}{2}+\delta+2\epsilon}\right)+O\left(T^{\epsilon}\right).$$

Since ϵ and δ can be any positive numbers, it follows that $M(x)=O\left(x^{\frac{1}{2}+\epsilon}\right)$ for all $\epsilon>0$.

□

Theorem 1.11. *If* $M(x)=O\left(x^{\frac{1}{2}+\epsilon}\right)$ *for all* $\epsilon>0$ *then the Riemann Hypothesis is true.*

Proof. Since $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$, there exist real numbers c, k such that $|M(x)| \leq c \cdot x^{\frac{1}{2}+\epsilon} \ \forall\, x \geq k$. We may write

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n<k} \frac{\mu(n)}{n^s} + \sum_{n \geq k} \frac{\mu(n)}{n^s}.$$

Clearly, the finite sum converges, and

$$\left| \sum_{n \geq k} \frac{\mu(n)}{n^s} \right| = O\left(n^{\frac{1}{2}+\epsilon-\sigma}\right)$$

But we may take $\sigma = \frac{1}{2} + \delta$ for some $\delta > 0$. Then we have $O\left(n^{\epsilon-\delta}\right)$. Since this must hold for all positive ϵ, we take $\epsilon < \delta$ and the O-terms $\to 0$ as $n \to \infty$ □

Corollary 1.12. *The Riemann Hypothesis is true iff* $\det(A_n) = O\left(n^{\frac{1}{2}+\epsilon}\right)$ *for all* $\epsilon > 0$.

References

[Ti] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function.*

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