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A New Stochastic Model for Systems Under General Repairs

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***Abstract*—Numerous stochastic models for repairable systems have been developed by assuming different time trends, and repair effects. In this paper, a new general repair model based on the repair history is presented. Unlike the existing models, the closed-form solutions of the reliability metrics can be derived analytically by solving a set of differential equations. Consequently, the confidence bounds of these metrics can be easily estimated. The proposed model, as well as the estimation approach, overcomes the drawbacks of the existing models. The practical use of the proposed model is demonstrated by a much-discussed set of data. Compared to the existing models, the new model is convenient, and provides accurate estimation results.**

***Index Terms*—Closed-form solution, confidence bounds, general repair, maximum likelihood estimation, proportional failure intensity, repairable system, virtual age.**

ACRONYMS[1]

HPP	Homogeneous Poisson Process
NHPP	Non-Homogeneous Poisson Process
PI	Proportional Intensity
MLE	Maximum Likelihood Estimate
LR	Likelihood Ratio
MTBF	Mean Time Between Failures

NOTATION

$N(t)$	cumulative number of failures up to time t
$m(t)$	$E[N(t)]$
$\lambda(t)$	failure intensity function
$\lambda_0(t)$	baseline failure intensity function
$\lambda_c(t)$	$m(t)/t$, expected value of cumulative failure intensity
t_i, x_i	time of the ith failure, and the ith inter-arrival time
q	repair effective factor in Kijima models
v_i	virtual age after the ith repair
a, b, γ	parameters of the proposed model
$R(t_i \vert t_{i-1})$	conditional reliability before the ith failure
$f(t_i \vert t_{i-1})$	conditional pdf of the ith failure time
$z(t)$	vector of functions of time t, and/or failure/repair history
$\wedge$	estimate

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[1]The singular and plural of an acronym are always spelled the same.

I. INTRODUCTION

FOR a repairable system, repair actions can bring the system to one of the following states: "as good as new" (perfect repair), "as bad as old" (minimal repair), or "better than old, but worse than new" (general repair).[2] Many stochastic models for repairable systems have been developed by assuming different repair effects. The vast majority of these models consider only the first two cases [3], [5], [17]. Recently, general repairs have received much attention [1], [2], [8], [10], [11], [13], [20], [21].

Specifically, the current research has shown significant interest in two particular approaches in modeling general repairs. The first one is called the proportional intensity (PI) model. In this subset, the repair effect is expressed by a reduction of the system failure intensity [13]. The second one is called the virtual age model in which the repair effect is expressed by a reduction of the system age [7], [9]. For these two subsets, many models have been developed, and methods for parameter estimation and goodness-of-fit tests discussed [10], [13], [15]. These models are statistically sound, but they are difficult to use in solving engineering problems due to their mathematical complexity. Especially, for the existing models, the closed-form solutions of reliability metrics of the system, such as the expected number of failures, instantaneous failure intensity, and mean time between failures (MTBF), are not available. It is mainly because these models are expressed in discrete forms, even though numerical solutions of the model parameters can be obtained through statistical inference procedures, such as the Maximum Likelihood approach. As a result, the current research effort highly relies on Monte Carlo simulation in order to obtain these reliability metrics [2], [10]. Moreover, little research has been done on estimating the confidence bounds of these reliability metrics. This also stems from the lack of closed-form solutions.

In this paper, a new model for general repairs based on repair history is proposed. The new model is expressed in a continuous form, and considers the repair effects and the time trends simultaneously. More importantly, unlike the existing models,

[2]The states which remain but which we do not treat in this paper are "better than new", and "worse than old".

this model can give the closed-form solutions for all the reliability metrics by solving a set of differential equations. Consequently, the confidence bounds of these reliability metrics can be obtained through the Fisher Information Matrix. The proposed method overcomes the drawbacks of the existing models for the analyses of complex repairable systems.

The remainder of this paper is organized as follows. Section II briefly reviews the existing repair models, and addresses their drawbacks. Section III presents the new model, and its properties. Section IV develops the statistical inference procedure for estimating the model parameters and confidence bounds of the reliability metrics of interest. In Section V, a numerical example is provided to demonstrate the use of the proposed model in maintenance practices. Section VI draws conclusions, and discusses the future research.

II. Overview of Models for Repairable Systems

The non-homogeneous Poisson process (NHPP) is one of the most popular models for repairable systems. Many models have been developed based on this process, such as the Crow-AMSAA model [5], and bounded intensity process (BIP) model [19]. These models account for the time trend effect, but can only describe the "as bad as old" type of repair processes. To address general repairs, Kijima virtual age models [11], [12], and PI models [3], [4], [13] have been proposed. The virtual age, and PH models are capable of modeling a vast variety of repairs. We will first review some of these models.

A. Non-Homogeneous Poisson Process

NHPP models are widely used to describe failure processes exhibiting certain trends, such as reliability growth, or deterioration. Let $N(t)$ be the cumulative number of failures up to time t (counting process), $m(t)$ be its expected value, and $\lambda(t)$ be the failure intensity, then the probability that $N(t)$ equals n $(n = 0, 1, 2, \ldots)$ is given by

$$\Pr\{N(t) = n\} = \frac{[m(t)]^n}{n!} e^{-m(t)}, \tag{1}$$

and $m(t)$ is

$$m(t) = \int_0^t \lambda(\tau) d\tau. \tag{2}$$

Functions such as the power law function [5], log-linear function [3], [4], [13], exponential function [19], and many others [17] have been utilized to describe the trend of failure intensity. Particularly, the log-linear function

$$\lambda(t) = e^{a+bt} \tag{3}$$

with the parameters a, and b, has been widely used in modeling NHPP due to its significant flexibility. If $b > 0$, the system will exhibit an increasing failure intensity; if $b < 0$, the system will have a decreasing failure intensity; and when $b = 0$, the failure process becomes a Homogeneous Poisson Process (HPP) with a constant failure intensity. In this paper, the log-linear function will be considered in our model development.

In a repair model following a NHPP, the time trend effect has been considered, but the general repair effect is essentially ignored. In practice, repair actions will bring the reliability indices of the system, e.g., the failure intensity, to somewhere between the "as good as new", and the "as bad as old". To consider the general repair effect, virtual age models, and PI models have been developed.

B. Virtual Age Model

Kijima [11], [12] developed two general repair models by introducing the virtual age concept. Consider a repairable system under instant repairs. Let $t_1, t_2, \ldots,$ be the successive failure times $(t_0 = 0)$, and $x_1, x_2, \ldots,$ be the inter-arrival times between failures such that

$$x_i = t_i - t_{i-1}, \quad i = 1, 2, \cdots \tag{4}$$

Denote the real age of the system by $t_n = \sum_{i=1}^n x_i$, the repair effective factor by q $(0 \le q \le 1)$, and the system virtual age after the nth repair by v_n $(v_0 = 0$ for a new system). In the literature, two famous virtual age models have been reported.

Kijima Model I:
Kijima model I assumes that the nth repair cannot remove the damage incurred before the $(n-1)$th failure; instead it can only reduce the additional age of the system partially from x_n to qx_n. Accordingly, the virtual age after the nth repair becomes

$$v_n = v_{n-1} + qx_n, \tag{5}$$

thus

$$v_n = q(x_1 + x_2 + \cdots + x_n) = qt_n. \tag{6}$$

Kijima Model II:
Suppose that, before the nth repair, the virtual age is $v_{n-1} + x_n$. Kijima model II assumes that the nth repair will remove the cumulative damage from both current, and previous failures; thus the virtual age after the repair becomes

$$v_n = q(v_{n-1} + x_n), \tag{7}$$

thus

$$v_n = q(q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n). \tag{8}$$

The basic idea of the virtual age models is to address the repair process as a *generalized renewal process*, and substitute the real time with the virtual age for calculations. For example, denote the virtual age at a given time t by $v(t)$, and consider the log-linear formulation, then the corresponding failure intensity becomes

$$\lambda(t) = e^{a+bv(t)} \tag{9}$$

Essentially, the Kijima virtual age models assume that repairs do not change the system's distinct probabilistic structure (the system lifetime distribution, and parameter structure), and the only change made by repairs is the system virtual age. This assumption is physically intuitive, and mathematically useful; however, as for the computation effort, these models highly rely on Monte Carlo simulation to obtain the reliability metrics. Thus, the practical application of these general repair models becomes difficult. More clearly, the substitution with the virtual age, such as in (9), raises the following problems in maintenance practices:

- For a future time t, the virtual age $v(t)$ is unknown. Therefore, the corresponding failure intensity $\lambda(t)$ is unknown, and so is the expected cumulative number of failures $m(t)$. Because there are no closed-form solutions of $\lambda(t)$ and $m(t)$, it is difficult to predict their future values, and give their confidence bounds. The only way to obtain the predicted values, and their confidence bounds, is to conduct simulation. However, simulation is time-consuming, and the accuracy of the results is mainly determined by the simulation scale.
- Because the expected value of the cumulative failure rate is $\lambda_c(t) = m(t)/t$, and the cumulative MTBF is $t/m(t)$, the same problems in (a) also exist for $\lambda_c(t)$, and the cumulative MTBF.

C. Proportional Intensity Model

It is clear that the virtual age models do not change the functional form of the failure intensity, but shift the intensity "block" horizontally along the time axis. In contrast, the PI models assume that repairs do not change the form of the baseline failure intensity, but shift the intensity "block" vertically along the intensity direction. To reflect such vertical shift (repair effects), the general PI model is

$$\lambda(t) = \lambda_0(t) \exp\left(\theta' z(t)\right), \tag{10}$$

where $\lambda_0(t)$ is the baseline model, $z(t)$ is the vector of functions that may depend upon both system operating time t and system operating/failure/repair history, and θ is the vector of unknown parameters.

Lawless & Thiagarajah [13] used a specific PI model of a modulated renewal process [4]:

$$\lambda(t) = \exp\left[\theta' z(t)\right] = \exp\left(\alpha + \beta t + \gamma\left(t - t_{N(t-)}\right)\right) \tag{11}$$

where α, β, γ are model parameters, and $t_{N(t-)}$ is the time of the latest failure prior to time t. Obviously, when this model is used for prediction, $t_{N(t-)}$ can be obtained only by simulation.

III. Model Development

A. Mathematical Formulation

Maintenance of a deteriorating system is often imperfect, with the condition of the system after maintenance being at a level somewhere between new, and its prior condition. Therefore, a general repair model is more realistic for describing the practical maintenance effort. In addition, it has been recognized that the chronological age, quite often, does not reflect the characteristic behavior of the system. To recognize the intrinsic characteristic of the system, entire information about the usage & repair history needs to be considered.

It seems plausible to assume that the cumulative number of repairs or failures is a useful metric that captures the age, use behavior, operating conditions, as well as repair history of the system. Motivated by this assumption, we propose a new general repair model based on the expected cumulative number of repairs (failures) as

$$\lambda(t) = \lambda_0(t) \exp\left[\theta' z(t)\right] = \lambda_0(t) \exp\left[\gamma m(t)\right] \tag{12}$$

where $\lambda_0(t)$ is the baseline failure intensity function, and the repair effect is reflected by the second term $\exp[\gamma m(t)]$ with parameter γ. By considering the effective repairs, i.e., the system reliability is recovered or at least not worsened after each repair, we have $\gamma \leq 0$. Note that when $\gamma = 0$ the proposed model includes the minimal repair model as a special case. In the following, we only focus on the discussion when $\gamma < 0$.

For this model, the closed-form solution of $m(t)$ can be obtained by solving the following differential equation:

$$\begin{aligned} \lambda(t) &= \frac{dm(t)}{dt} = \lambda_0(t) \exp\left[\gamma m(t)\right] \\ \Rightarrow \qquad m(t) &= -\frac{1}{\gamma} \ln\left[-\gamma\left(\int \lambda_0(t)dt + C\right)\right] \end{aligned} \tag{13}$$

where the constant C can be obtained from the boundary condition $m(0) = 0$. Specifically, if the log-linear baseline intensity function is used, i.e., $\lambda_0(t) = e^{a+bt}$, the new model becomes

$$\lambda(t) = e^{a+bt+\gamma m(t)} \tag{14}$$

We will focus on this specific model hereafter. Following (13), the new model becomes

$$m(t) = -\frac{1}{\gamma} \ln\left[-\gamma\left(\frac{1}{b}e^{a+bt} + C\right)\right] \tag{15}$$

Then, from $m(0) = 0$, the constant C can be obtained as $C = -1/\gamma - (1/b)e^a$, when $b \neq 0$. Therefore, substituting C into (15) yields

$$m(t) = -\frac{1}{\gamma} \ln\left[-\frac{\gamma}{b}e^{a+bt} + 1 + \frac{\gamma}{b}e^a\right], \quad b \neq 0 \tag{16}$$

For some components or systems, there is no time trend (aging); the total failure intensity is only affected by the number of bugs found during a debugging process. For this case, we have $b = 0$, which leads to a constant baseline failure intensity. Resolving (13) yields

$$m(t) = -\frac{1}{\gamma} \ln\left[-\gamma\left(e^a t - \frac{1}{\gamma}\right)\right], \quad b = 0 \tag{17}$$

Equations (16) & (17) are closed-form solutions of $m(t)$, and its confidence bounds can be calculated from the variance-covariance matrix of the maximum likelihood estimates (MLE) of

a, b, and γ. Furthermore, taking the first derivative of $m(t)$ with respect to time t yields

$$\lambda(t) = e^{a+bt+rm(t)} = \frac{be^{a+bt}}{b - \gamma e^{a+bt} + \gamma e^{a}}, \quad b \neq 0 \tag{18}$$

and

$$\lambda(t) = e^{a+rm(t)} = \frac{e^{a}}{1 - \gamma e^{a} t}, \quad b = 0 \tag{19}$$

Then, the confidence bounds of $\lambda(t)$ can be calculated based on the variance-covariance matrix of the parameter estimates of a, b, and γ as well.

B. Model Properties

The properties of system failure intensity over time are indicative of the effectiveness of performing repairs.

First, the monotonic properties of the new model are investigated. When $b \neq 0$, taking the derivative of $\lambda(t)$in (18) yields

$$\frac{d\lambda(t)}{dt} = \frac{b^2 e^{a+bt}(b + \gamma e^{a})}{(b - \gamma e^{a+bt} + \gamma e^{a})^2} \tag{20}$$

Because $b^2 e^{a+bt} > 0$, it is sufficient to study the sign of $b + \gamma e^{a}$. Obviously, when $b + \gamma e^{a} = 0$, the model represents a homogeneous Poisson process (HPP) with the constant failure intensity of $\lambda(t) = e^{a}$. This equation means the repair effect, and the system wear-out are canceling each other. When $b + \gamma e^{a} > 0$, the system exhibits an increasing trend in failure intensity, while it has a decreasing trend when $b + \gamma e^{a} < 0$. Similarly, when $b = 0$, taking the derivative of $\lambda(t)$ in (19) yields

$$\frac{d\lambda(t)}{dt} = \frac{\gamma e^{2a}}{(1 - \gamma e^{a} t)^2} \tag{21}$$

Because $\gamma < 0$, $\lambda(t)$ will be decreasing over time. In other words, the reliability of the system is improved due to repairs.

As for the asymptotic properties of the model, when $b \neq 0$, from (18), if $t \to \infty$,

$$\lim_{t\to\infty} \lambda(t) = \lim_{t\to\infty} \frac{be^{a+bt}}{b - \gamma e^{a+bt} + \gamma e^{a}} = \begin{cases} -\frac{b}{\gamma} & \text{when } b > 0 \\ 0 & \text{when } b < 0 \end{cases} \tag{22}$$

Obviously, when $b < 0$, the failure probability of the system will approach 0 as $t \to \infty$; otherwise, it will eventually have the same failure behavior as a HPP with the constant failure intensity of $-b/\gamma$. Likewise, when $b = 0$, from (19), if $t \to \infty$, $\lim_{t\to\infty} \lambda(t) = 0$. One example in this case is the software failure & debugging process, as mentioned previously.

The following analyses will focus on the case when $b \neq 0$, while similar procedures can be applied when $b = 0$.

IV. Statistical Inference Procedure

There are many approaches to estimate the model parameters from historical failure data. In this paper, the Maximum Likelihood approach is utilized.

A. Maximum Likelihood Estimates

For the new model in (14), the empirical failure intensity is

$$\lambda(t) = e^{a+bt+\gamma N(t)} \tag{23}$$

which is the empirical sample of the process. In other words, the continuous process with $\lambda(t) = e^{a+bt+\gamma m(t)}$ is used to approximate the discontinuous process with $\lambda(t) = e^{a+bt+\gamma N(t)}$. Therefore, the conditional reliability before the ith failure is

$$R(t_i|t_{i-1}) = e^{-\frac{1}{b}\left(e^{a+bt_i+(i-1)\gamma} - e^{a+bt_{i-1}+(i-1)\gamma}\right)}, \tag{24}$$

and the conditional pdf of the ith failure time is

$$f(t_i|t_{i-1}) = e^{a+bt_i+(i-1)\gamma - \frac{1}{b}\left(e^{a+bt_i+(i-1)\gamma} - e^{a+bt_{i-1}+(i-1)\gamma}\right)} \tag{25}$$

So, the likelihood function of the observed data is

$$L(\text{Data}|a,b,\gamma) = \prod_{i=1}^{n} f(t_i|t_{i-1}), \tag{26}$$

and the corresponding log-likelihood function is

$$\begin{aligned} &\ln\left(L(\text{Data}|a,b,\gamma)\right) \\ &= \sum_{I=1}^{n} (a + bt_i + (i-1)\gamma \\ &\qquad - \frac{1}{b}\left(e^{a+bt_i+(i-1)\gamma} - e^{a+bt_{i-1}+(i-1)\gamma}\right)\Big) \\ &= an + \frac{n(n-1)}{2}\gamma + b\sum_{i=1}^{n} t_i - \frac{1}{b}e^{a}\sum_{i=1}^{n} e^{bt_i+(i-1)\gamma} \\ &\quad + \frac{1}{b}e^{a}\sum_{i=1}^{n} e^{bt_{i-1}+(i-1)\gamma} \end{aligned} \tag{27}$$

From (24), the first derivatives of the log-likelihood function with respect to the parameters are given by

$$\begin{aligned} \frac{\partial \ln(L)}{\partial a} &= n - \frac{1}{b}e^{a}\sum_{i=1}^{n} e^{(i-1)\gamma+bt_i} \\ &\quad + \frac{1}{b}e^{a}\sum_{i=1}^{n} e^{(i-1)\gamma+bt_{i-1}} \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial \ln(L)}{\partial b} &= \sum_{i=1}^{n} t_i + \frac{1}{b^2}e^{a}\sum_{i=1}^{n} e^{(i-1)\gamma+bt_i} \\ &\quad - \frac{1}{b}e^{a}\sum_{i=1}^{n}\left(e^{(i-1)\gamma+at_i}t_i\right)\frac{1}{b^2}e^{a}\sum_{i=1}^{n} e^{(i-1)\gamma+bt_{i-1}} \\ &\quad + \frac{1}{b}e^{a}\sum_{i=1}^{n}\left(e^{(i-1)\gamma+bt_{i-1}}t_{i-1}\right) \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{\partial \ln(L)}{\partial \gamma} &= \frac{n(n-1)}{2} - \frac{1}{b}e^{a}\sum_{i=1}^{n}\left(e^{(i-1)\gamma+bt_i}(i-1)\right) \\ &\quad + \frac{1}{b}e^{a}\sum_{i=1}^{n}\left(e^{(i-1)\gamma+bt_{i-1}}(i-1)\right) \end{aligned} \tag{30}$$

The MLE of a, b, and γ can be obtained by setting (25)–(27) to zero, and solving the equations simultaneously. Then, the MLE of $m(t)$, and $\lambda(t)$ become

$$\hat{m}(t) = -\frac{1}{\hat{\gamma}} \ln\left(-\frac{\hat{\gamma}}{\hat{b}} e^{\hat{a}+\hat{b}t} + 1 + \frac{\hat{\gamma}}{\hat{b}} e^{\hat{a}}\right), \tag{31}$$

and

$$\hat{\lambda}(t) = \frac{\hat{b} e^{\hat{a}+\hat{b}t}}{\hat{b} - \hat{\gamma} e^{\hat{a}+\hat{b}t} + \hat{\gamma} e^{\hat{a}}} \tag{32}$$

B. Variance-Covariance Matrix of Parameter Estimates

Based on the asymptotic theory for MLE, the variance-covariance matrix of the MLE $\hat{a}$, $\hat{b}$, and $\hat{\gamma}$ can be obtained by taking the inverse of the Fisher Information Matrix [14] evaluated at the MLE as

$$\begin{pmatrix} \text{var}[\hat{a}] & \text{Cov}[\hat{a},\hat{b}] & \text{Cov}[\hat{a},\hat{\gamma}] \\ \text{Cov}[\hat{a},\hat{b}] & \text{var}[\hat{b}] & \text{Cov}[\hat{b},\hat{\gamma}] \\ \text{Cov}[\hat{a},\hat{\gamma}] & \text{Cov}[\hat{b},\hat{\gamma}] & \text{var}[\hat{\gamma}] \end{pmatrix} = \begin{pmatrix} -\frac{\partial^2 \ln(L)}{\partial a^2} & -\frac{\partial^2 \ln(L)}{\partial a \partial b} & -\frac{\partial^2 \ln(L)}{\partial a \partial \gamma} \\ -\frac{\partial^2 \ln(L)}{\partial a \partial b} & -\frac{\partial^2 \ln(L)}{\partial b^2} & -\frac{\partial^2 \ln(L)}{\partial \gamma \partial b} \\ -\frac{\partial^2 \ln(L)}{\partial a \partial \gamma} & -\frac{\partial^2 \ln(L)}{\partial \gamma \partial b} & -\frac{\partial^2 \ln(L)}{\partial \gamma^2} \end{pmatrix}^{-1}_{a=\hat{a};b=\hat{b};\gamma=\hat{\gamma}} \tag{33}$$

C. Confidence Bounds of $m(t)$, and $\lambda(t)$

From (17), the partial derivatives of $m(t)$ with respect to the model parameters are

$$\frac{\partial m(t)}{\partial a} = -\frac{1}{\gamma}\frac{1}{B_1}\frac{\gamma}{b}(-B) = \frac{B}{B_1 b} \tag{34}$$

$$\frac{\partial m(t)}{\partial b} = -\frac{1}{B_1}\left[\frac{1}{b^2}B - \frac{t}{b}e^{a+bt}\right] \tag{35}$$

$$\frac{\partial m(t)}{\partial \gamma} = \frac{1}{\gamma^2}\ln B_1 + \frac{1}{\gamma}\frac{1}{B_1}\frac{1}{b}B \tag{36}$$

where $B = e^{a+bt} - e_a$; $B_1 = 1 + (\gamma/b)(e^a - e^{a+bt})$. Using the delta method [16], the variance of $\hat{m}(t)$ can be obtained as

$$\begin{aligned} \text{var}(\hat{m}(t)) = & \left(\frac{\partial m(t)}{\partial b}\right)^2 \text{var}(\hat{b}) + \left(\frac{\partial m(t)}{\partial a}\right)^2 \text{var}(\hat{a}) \\ & + \left(\frac{\partial m(t)}{\partial \gamma}\right)^2 \text{var}(\hat{\gamma}) \\ & + 2\left(\frac{\partial m(t)}{\partial b}\right)\left(\frac{\partial m(t)}{\partial a}\right)\text{Cov}(\hat{b},\hat{a}) \\ & + 2\left(\frac{\partial m(t)}{\partial \gamma}\right)\left(\frac{\partial m(t)}{\partial a}\right)\text{Cov}(\hat{\gamma},\hat{a}) \\ & + 2\left(\frac{\partial m(t)}{\partial b}\right)\left(\frac{\partial m(t)}{\partial \gamma}\right)\text{Cov}(\hat{b},\hat{\gamma}) \end{aligned} \tag{37}$$

where the partial derivatives are evaluated at the MLE $\hat{a}$, $\hat{b}$, and $\hat{\gamma}$. By assuming the log transformation of $\hat{m}(t)$ follows the s-normal distribution, the approximate $100(1-\alpha)\%$ confidence bounds [14] of $m(t)$ are given by

$$[m_L(t) m_U(t)] = [\hat{m}(t)/w \quad \hat{m}(t) \times w] \tag{38}$$

where $w = \exp[z_{1-\alpha/2}\sqrt{Var(\hat{m}(t))}/\hat{m}(t)]$, and $z_{1-\alpha/2}$ is the $1-\alpha/2$ percentile of the standard s-normal distribution. Similarly, the confidence bounds of $\lambda(t)$ in (18) can also be calculated using this method, which will not be repeated here.

TABLE I
FAILURE TIME DATA FROM PLANE 7 (IN CUMULATIVE OPERATING HOURS)

97	304	579	782	1155	1567	1857
148	322	659	988	1371	1630	2020
159	434	660	1070	1417	1648	2044
163	502	676	1124	1528	1839	

D. Likelihood Ratio Tests for Time Trend, and Repair Effect

The likelihood ratio (LR) test is a statistical test of the goodness-of-fit between two hierarchically nested models. Especially, it can be implemented to check the significance of adding additional parameters in a model. Let $\underline{\theta}_k$ be a vector consisting of k additional model parameters. The LR statistic $\Lambda = -2\ln(\sup\{L_{\underline{0}_k} : \underline{\theta}_k = \underline{0}_k\}/\sup\{L_{\underline{\theta}_k} : \underline{\theta}_k \neq \underline{0}_k\})$ follows the χ^2_k distribution with k degrees of freedom, where $\sup\{L_{\underline{0}_k} : \underline{\theta}_k = \underline{0}_k\}$, and $\sup\{L_{\underline{0}_k} : \underline{\theta}_k \neq \underline{0}_k\}$ are the maximum likelihood values obtained for $\underline{\theta}_k = \underline{0}_k$, and $\underline{\theta}_k \neq \underline{0}_k$ respectively. It follows that the null hypothesis, $\text{H}_0 : \underline{\theta}_k = \underline{0}_k$, is rejected if $\Lambda > \chi^2_{k,\alpha}$, in which $\chi^2_{k,\alpha}$ is the upper α quantile of the χ^2_k distribution with k degrees of freedom. Because the significance of a specific effect reflected by the data can be measured by the value of the associated parameter in the repair model, it is equivalent to check the significance of the model parameter using the LR test as demonstrated next.

V. NUMERICAL EXAMPLE

The proposed model is demonstrated using the well known data on airplane air-conditioning failures [18]. One data set denoted as Plane 7 [3], [13] is analysed. Table I gives the failure times in cumulative operating hours. The baseline failure intensity $\lambda_0(t) = e^{a+bt}$ is assumed for the new model.

A. Tests for Time Trend, and Repair Effect

First of all, the likelihood ratio approach presented in Section IV is utilized in testing for the time trend, and the repair effect. The first hypothesis test is expressed as

- H_0 Neither time nor repair effect exists ($[b,\gamma] = [0,0]$).
- H_1 At least one effect exists (either $b \neq 0$, or $\gamma \neq 0$, or both).

which tests the time trend, and the repair effect simultaneously. Under the null hypothesis H_0, by setting $b = 0$, and $\gamma = 0$, the maximum log-likelihood value is obtained as -143.8356. On the other hand, the maximum log-likelihood value under the full model is -138.9524. Then, the test statistic equals $\Lambda = 9.7664$, which is greater than $\chi^2_{2,0.05} = 5.99$, the chi-square value with

TABLE II
PARAMETER ESTIMATES OF DIFFERENT MODELS

Model parameter	NHPP	Kijima I	Kijima II	New model
b	-9.06E-05	0.00441	0.00647	0.00891
a	-4.23568	-4.63138	-5.19569	-3.78046
q	1	0	0.49397	———
γ	———	———	———	-0.70700
$Ln\,(L)$	-143.78580	-143.04068	-142.01532	-138.95240

two degrees of freedom at the significance level 0.05. Therefore, the HPP is rejected, implying either the time trend effect, or the repair effect, or both exist. To study the time trend effect individually, the following test is conducted as

H_0 No time trend exists ($b = 0$).

H_1 Time trend exists ($b \neq 0$).

By setting $b = 0$, the maximum log-likelihood value is obtained as -206.0910. The test statistic is $\Lambda = 134.2772$, which is much greater than $\chi^2_{1,0.05} = 3.84$. Therefore, H_0 is rejected, meaning that the time trend is significant. Similarly, the third test can be implemented to check the significance of the repair effect by testing

H_0 No repair effect exists ($\gamma = 0$).

H_1 Repair effect exists ($\gamma \neq 0$).

In this test, the maximum log-likelihood value equals -143.7858. The test statistic is $\Lambda = 9.6668$, which is greater than $\chi^2_{1,0.05} = 3.84$, implying that the repair effect exists. These tests show the evidence of both the time trend, and the repair effect; therefore the full model will be utilized for the subsequent analyses. Note that Lawless & Thiagarajah [13] drew different conclusions regarding the trend, and repair effect for the same data. However, by comparing the log-likelihood values in both works, we can see that the values we obtained are higher (e.g., in testing of H_0: No repair effect exists ($\gamma = 0$), the value is -143.7858 in our work, whereas the value is -144.20 in the last column of Table I in [13]). In terms of MLE, the parameter estimates we obtained are more accurate; so our conclusions may be more appropriate in capturing the trend, and the repair effect.

B. Model Comparison

For comparison, another three repair models, NHPP, Kijima I, and Kijima II models are considered. In the NHPP model, the failure intensity is set to be $\lambda(t) = e^{a+bt}$; and for Kijima I, and II models $\lambda(t) = e^{a+bv(t)}$. The parameter estimates of these models are summarized in Table II. The parameters of all the models are estimated using the Maximum Likelihood Estimation method. The likelihood functions of NHPP, and Kijima I are given in [5] & [22]. The likelihood function of Kijima II is similar to Kijima I; the only difference is the way of calculating the virtual age. From Table II, it can be seen that by coincidence the Kijima I model becomes a perfect renewal model because $q = 0$. Moreover, the NHPP model shows a decreasing failure intensity as $b < 0$, while the others exhibit an increasing trend of failure intensity before repairs are conducted, i.e. $d\lambda_0(t)/dt > 0$. Note that the repair effect or the repair trend is reflected by the parameters q, and γ. More specifically, the NHPP assumes $q = 1$, i.e., there is no reliability improvement after repairs. The Kijima I model shows that the system is as good as new after each repair. As for the Kijima II, and the new model, both values show that repairs reduce the system failure intensity, and repair actions restore the system back to somewhere between the "as good as new," and the "as bad as old" conditions.

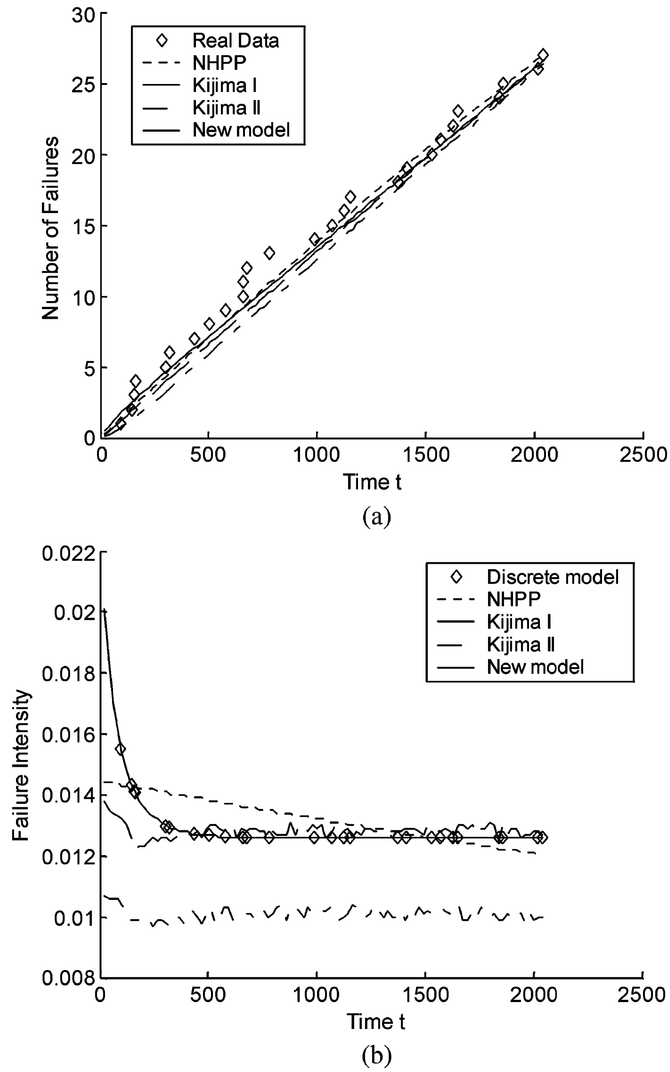

Fig. 1. Estimates of (a) the expected number of failures $m(t)$, and (b) the failure intensity $\lambda(t)$.

For comparison, Fig. 1 gives the estimates of the expected number of failures $m(t)$, and the failure intensity $\lambda(t)$, using the different models. From Fig. 1(a), we can see that the prediction errors of the Kijima models are large, while both the new model and the NHPP model appear to fit the data better. The only difference between the new model and the NHPP model is whether or not the repair effect is considered, which has already been tested before. In Fig. 1(b), the discrete model for the instantaneous failure intensity is expressed as $\lambda(t) = e^{a+bt+\gamma N(t)}$, and represented by the discrete diamond marks. The figure shows that the solid line goes through the diamond marks, meaning that the proposed continuous model in (14) fits the empirical discrete model in (23) quite well.

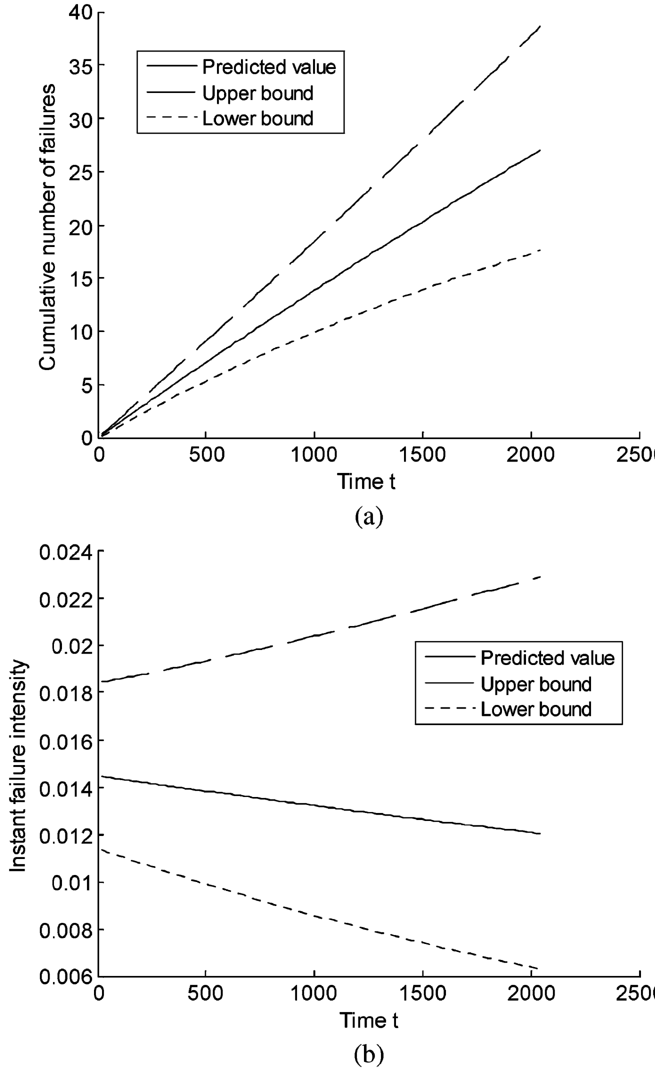

Fig. 2. Predicted (a) failure number, and (b) instantaneous failure intensity with 95% confidence bounds (2-sided)—NHPP model with log-linear intensity.

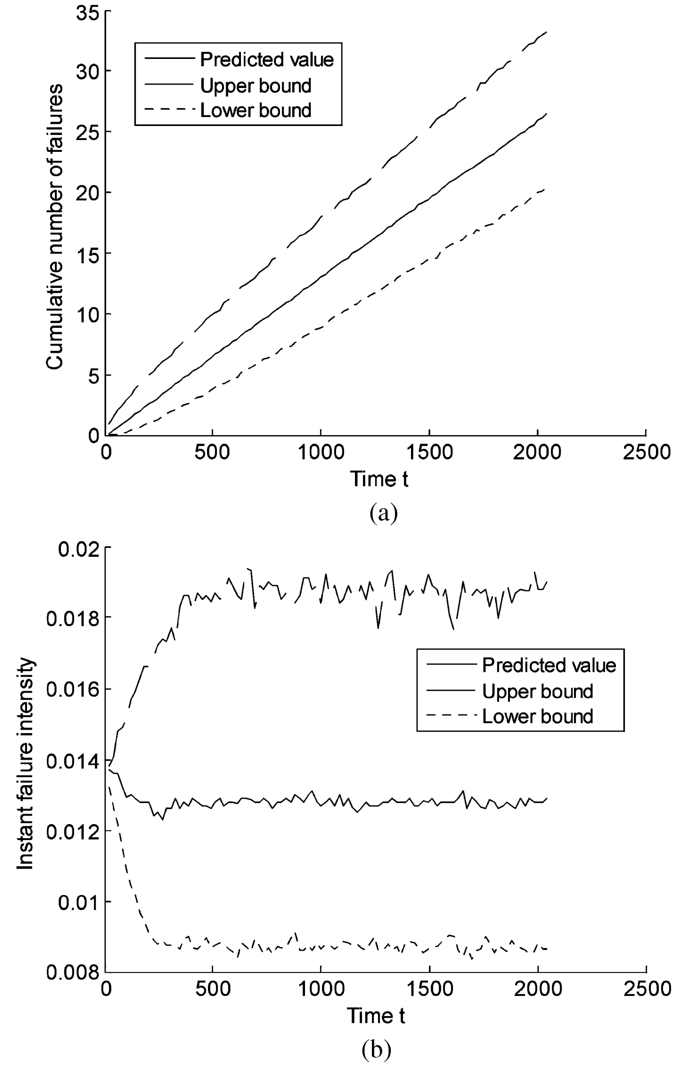

Fig. 3. Predicted (a) failure number, and (b) instantaneous failure intensity with 95% confidence bounds (2-sided)—Kijima I model with log-linear intensity.

The confidence bounds of the number of failures $m(t)$, and the failure intensity $\lambda(t)$, are given in Figs. 2–5. For the Kijima models, the figures are obtained by the Monte Carlo simulation approach because there are no closed-form solutions. From the figures, the advantages of the new model can be seen clearly. Although the NHPP model also gives smooth curves because it also has closed-form solutions, the confidence bounds are relatively wide. The Kijima models can provide similar prediction results to the new model; however, it can only give the prediction, and confidence bounds by simulation, which is time-consuming, and the quality of the prediction results highly depends on the simulation scale.

Moreover, it is worth recognizing the differences between the minimal repair model (see Fig. 2), and the general repair models (seeFigs. 3–5). In Fig. 2, the failure intensity, represented by a smooth curve, is monotonically decreasing. In Figs. 3 & 4, the failure intensity functions exhibit decreasing time trends first, and afterwards stay even around certain values. The oscillation components attached to the time trends stem from simulations involved in the Kijima models. For the new model, Fig. 5 gives the smooth failure intensity function, which decreases first, and quickly approaches a constant. Such a feature can be verified by examining the model parameters. From Table II, because $b \neq 0$, and $b + \gamma e^{a} = -0.00722 < 0$, the failure intensity will exhibit a decreasing trend as discussed in Section III; moreover, because $b < 0$, the failure intensity will eventually approach to the constant $-b/\gamma = 0.01260$. Note that this detailed feature in trend has not been captured in the analyses by Lawless & Thiagarajah [13].

For the NHPP, and the new model, analytical Fisher bounds are used, and the lognormal distribution is assumed. Therefore, they are approximation bounds, and the accuracy of the approximation is affected by the number of failures used in the model parameter estimation. In the given example, because we have 27 failures, it is reasonable to use the Fisher bounds.

VI. Conclusion

In this paper, a new general repair model based on the number of repairs has been proposed. The model considers the cumulative repair effects, and the time trends simultaneously. More importantly, unlike the existing general repair models, this model is analytically solvable for all the reliability metrics

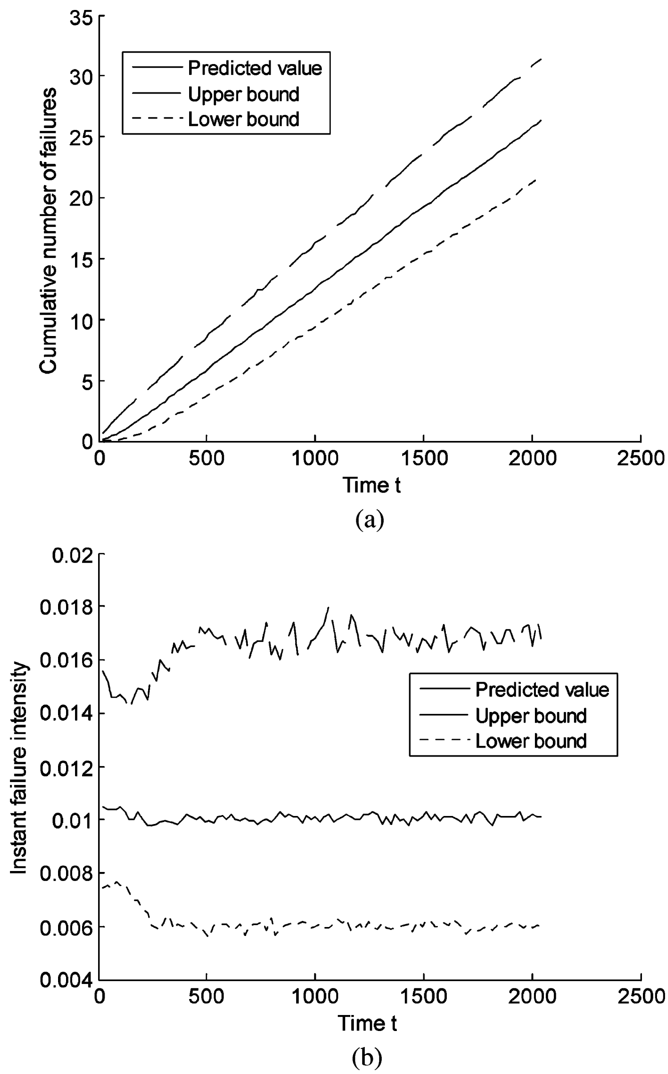

Fig. 4. Predicted (a) failure number, and (b) instantaneous failure intensity with 95% confidence bounds (2-sided)—Kijima II model with log-linear intensity.

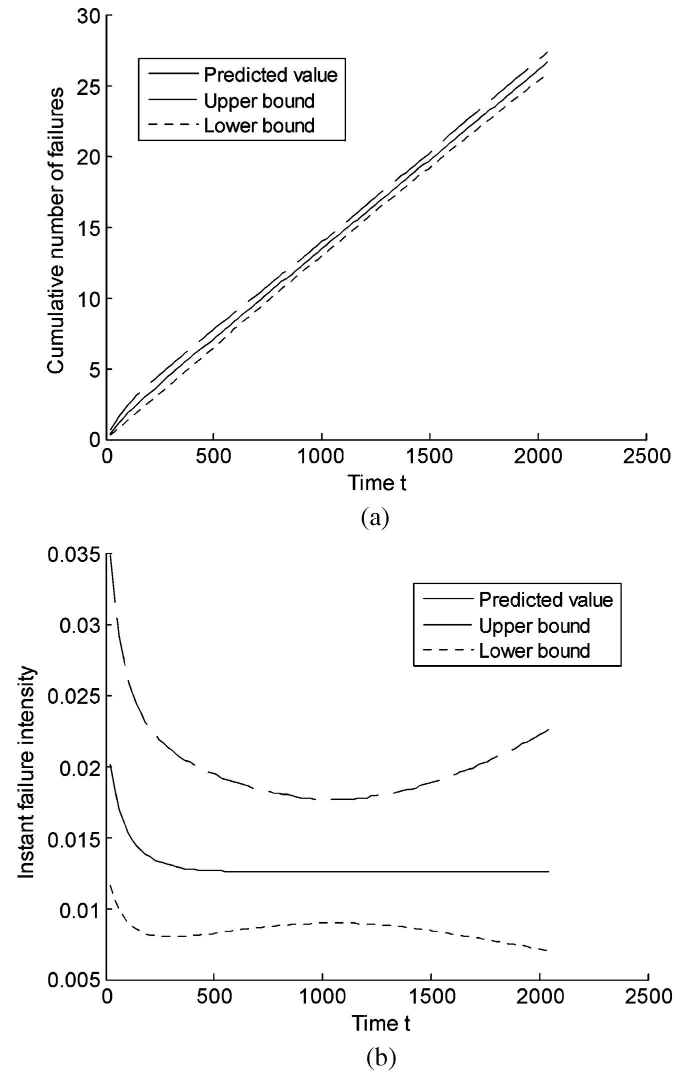

Fig. 5. Predicted (a) failure number, and (b) instantaneous failure intensity with 95% confidence bounds (2-sided)—new model with log-linear intensity.

by solving a set of differential equations. Consequently, the confidence bounds of these reliability metrics can be easily estimated. The proposed analytical method has strong statistical foundations, and overcomes all the drawbacks of the existing models for the analyses of complex repairable systems. Specifically, compared to the existing models, the proposed model is capable of giving faster, more accurate estimation results without relying on time-consuming Monte Carlo simulations. The numerical example shows that the proposed method is promising, and efficient. Furthermore, it holds promise both for being useful in real industrial applications, as well as having a structure permitting further generalizations of repairable system analyses.

Many extensions to the proposed model are possible, including the interval failure data analyses, use of general baseline failure intensity functions, modeling of the interaction between the time trend and repairs, and inclusion of repair costs, which would generate more practical interest. Moreover, the applications of the model in such areas as the human system reliability, healthcare, and software reliability, which have their specific failure intensity trends, and more complex "repair" effects, would bring about further extensions. Furthermore, some new research has been done on optimal repair allocation under limited resources (e.g., see [6]). Because the proposed model can capture both the time trend, and repair effects in an analytical, closed-form way, it has a great deal of potential for making optimal allocation when different repair or maintenance policies are available.

APPENDIX

Entries of the Fisher Information Matrix:

From (27), the second derivatives of the log-likelihood function are given by

$$
\begin{aligned}
\frac{\partial^2 \ln(L)}{\partial a^2} &= \frac{1}{b}(A_1 - A) \\
\frac{\partial^2 \ln(L)}{\partial b^2} &= \frac{2}{b^3}(A_1 - A) + \frac{2}{b^2}(AT - AT_1) \\
&\quad + \frac{1}{b}(ATT_1 - ATT)
\end{aligned}
$$

$$\frac{\partial^2 \ln(L)}{\partial \gamma^2} = -\frac{1}{b}ALL + \frac{1}{b}ALL_1 = \frac{1}{b}(ALL_1 - ALL)$$
$$\frac{\partial^2 \ln(L)}{\partial a \partial b} = \frac{1}{b^2}A - \frac{1}{b}AT - \frac{1}{b^2}A_1 + \frac{1}{b}AT_1$$
$$\frac{\partial^2 \ln(L)}{\partial a \partial \gamma} = \frac{1}{b}(AL_1 - AL)$$
$$\frac{\partial^2 \ln(L)}{\partial \gamma \partial b} = \frac{1}{b^2}(AL - AL_1) + \frac{1}{b}(ALT_1 - ALT)$$

where

$$A = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i}\right),$$
$$AT = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i} t_i\right),$$
$$ATT = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i} t_i^2\right),$$
$$A_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}}\right),$$
$$AT_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}} t_{i-1}\right),$$
$$ATT_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}} t_{i-1}^2\right),$$
$$AL = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i}(i-1)\right),$$
$$ALL = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i}(i-1)^2\right),$$
$$ALT = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_i}(i-1)t_i\right),$$
$$AL_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}}(i-1)\right),$$
$$ALL_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}}(i-1)^2\right), \quad \text{and}$$
$$ALT_1 = e^a \sum_{i=1}^{n} \left(e^{(i-1)\gamma + bt_{i-1}}(i-1)t_{i-1}\right).$$

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